KINEMATICS AND DYNAMICS

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ELEMENTARY EXPERIMENTAL MECHANICS

BY

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PREFACE

In this book an attempt is made to combine theory and practice as closely as possible. Success in teaching is in proportion to the extent to which the active initiative of the student is aroused, and nothing is so effective in this respect as laboratory work, if it be of the right kind. The use of the hand and the eye affords an invaluable stimulus to the imagination and the reason. Without such personal work the interest awakened by a good lecture is apt to be superficial and temporary, and the preparation insured by recitations is too often reluctant and unfruitful. Mechanics is the most fundamental and least attractive part of physics, and in the teaching of it lectures and recitations need all of the aid that laboratory work can supply.

A grasp of principles is of more value to the average student than skill in measurement. While the exercises in this book have been chosen chiefly with a view to the elucidation of principles, the need of an adequate degree of precision in the necessary measurements has been kept in mind. In most cases a test of the accuracy of the work is supplied by a comparison of the results of theory and experiment. The course is not a substitute for, but is preliminary to, a course in the more precise measurement of physical constants. *Its aim is to stimulate reflection on concepts and principles, and the value of each exercise is*

in proportion to the importance and number of the physical ideas which must be considered in performing the exercise. With a few exceptions the exercises have been tried by large classes of students and have been found satisfactory. The exceptions have been carefully tested by myself or an assistant. The introduction of numerous original exercises is due to a lack of suitable familiar experiments. Many well-known exercises have been omitted either because they do not strongly enforce mechanical concepts and principles or because they require complex or expensive instrumental means.

To serve the purpose stated above, each exercise should follow the related lecture or recitation as closely as possible; it will lose much of its value if postponed for several weeks. I have therefore endeavored to choose exercises that call for comparatively simple apparatus, so that sufficient copies of each part may be procured to enable all the students in a class (or section) to work simultaneously and separately on each experiment. (The practice of having two or more students work together is very unsatisfactory.) Important parts of the apparatus serve for a large number of exercises. A few experiments which require apparatus of greater complexity may, if necessary, be omitted or may be performed by the instructor in presence of the class, the calculations being left to the latter.

The statements of theory have of necessity been brief; but brevity in this respect is hardly to be regretted. Diffuseness and repetition are desirable in an oral explanation, but a printed statement can be reread until it is mastered. Diffuseness in a text-book often defeats its

own aim. Bright students skip prolix explanations, and others are often only puzzled and confused by what is unessential; statements of principles cannot be predigested by dilution. The directions for the experiments have not been made so full as to leave nothing to exercise the judgment of the student. Condensed formulæ for calculation and tabular forms for reporting have not been supplied. These often tempt the student to work blindly and confine his attention to finding figures to fit the formulæ and fill the blanks. The instructor may supply such as he thinks necessary either in the lecture which precedes the exercise or on the laboratory blackboard. The topics found under the heading "Discussion" must be regarded as mere suggestions; many questions will be suggested by the laboratory work or by the subsequent discussion between the class and the instructor. Illustrative experiments may be introduced in either lecture or discussion.

The apparatus is for the most part simple and readily constructed. It may also be obtained at very reasonable rates from the International Instrument Co. of Cambridge, Massachusetts.

I have to thank Dr. A. W. Ewell, Assistant Professor of Physics in the Worcester Polytechnic Institute, for valuable suggestions and assistance, and also Mr. C. F. Howe and Mr. C. B. Harrington, assistants in physics, for much valuable aid.

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ELASTIC SOLIDS AND FLUIDS

For a shorter course omit Exercises V–VIII, XV, XVI (2), XVIII, XIX, XXII, XXIII, XXVII, XXIX–XXXI, XXXVII, and §§ 32, 46–49, (proofs of formulæ in) 72 and 75, 76, (part of) 79, 87, 107, 114, 115, 119, 120, 130, 159, (part of) 160, 171, 177, 179.

KINEMATICS

CHAPTER I

UNITS AND MEASURING INSTRUMENTS

1. *Mechanics* is the science of motion and of the causes of changes in the motion of bodies. *Kinematics* is the branch of Mechanics which treats of motion. It is a preliminary to the branch which treats of the causes of change of motion, or *Dynamics.*

The ideas with which we deal in Kinematics are those of Geometry and Time. Geometrical relations are described by means of lengths of lines and magnitudes of angles, and time relations are described by means of intervals of time. To measure one of these we must compare it with a standard of its own kind, called a *unit.*

2. Units of Length. — The *metre* was intended by those who devised it to be equal to $\frac{1}{10000000}$ of the distance from the north pole to the equator, along the meridian through Paris. While this *derivation* of the metre is of historical interest, the metre is actually *defined* as being the distance between two parallel lines on a platinum-iridium bar kept at Sèvres, near Paris. Every metre scale is intended to be a copy, more or less accurate, of this standard. Submultiples of the metre are the decimetre (0.1 m.), the centimetre (0.01 m.), and the milli-

metre (0.001 m.). A multiple, the kilometre (1000 m.), is used for stating great lengths. The centimetre is the metric unit mostly used in Physics.

The *yard* is defined in the United States as $\frac{36}{39.37}$ of the Paris metre. In Great Britain it is defined as the distance between two lines on a bronze bar kept at the office of the Exchequer in London.

A submultiple of the yard, the foot ($\frac{1}{3}$ yd.), is the unit of length mostly used by engineers in English-speaking countries.*

3. Some Instruments used in measuring Lengths. — The *beam-compass* consists of a straight rod and two pointers movable along the rod. It is used in measuring a distance when a scale cannot be brought into position to make

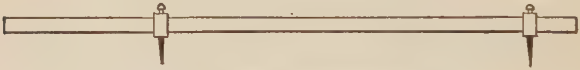

Fig. 1. — The Beam-compass.

the measurement directly. The points are adjusted until they coincide with the ends of the length to be measured. They are then clamped in that position on the rod and the distance between them measured by a scale.

An inexpensive beam-compass that will suffice for these experiments may be made from a brass rod about 30 cm. in length and two large-sized electrical "connectors." The bore of the connectors should slightly exceed the diameter of the rod. Large-sized sewing-needles inserted by the head into small holes drilled in the connectors and then soldered in position complete the instrument.

* For the ratios of metric and English units, see Table in Appendix.

A *vernier caliper* is essentially a beam-compass, the beam of which is graduated and provided with a device called a *vernier*, for accurately reading the fractions of the smallest division of the scale. The principle of the vernier will be understood from a study of Fig. 2. Each unit of the small scale, called the vernier, is (in the instrument figured) $\frac{1}{10}$ shorter than each unit of the main scale, or 10 vernier divisions equal 9 scale divisions. Let

FIG. 2.—The Vernier.

one end of the length to be measured fall between two of the small divisions of the main scale, say at A, between 5.1 and 5.2. Then it is easily seen that, if the third division of the vernier coincide with a division of the main scale, the distance from 5.1 to A is $\frac{3}{10}$ of the smallest division of the main scale. Hence the reading in this case is 5.13. In general, if the vernier be made so that n vernier divisions equal $n-1$ scale divisions, then the "least count" of the vernier will be $\frac{1}{n}$ of a scale division. To distinguish it from another type of vernier, the one just described is sometimes called a "direct" vernier. A "retrograde" vernier has a length equal to $n+1$ scale divisions divided into n parts on the vernier; its divisions are numbered in a direction opposite to that of the scale divisions. The two types of vernier are read in essentially the same way.

The vernier is very important in many practical measurements. The simplest way of mastering its principle is to make an attempt to construct one. (Exercise 1.)

In the *micrometer caliper* the device used for estimating fractions of the smallest scale division depends on the fact that when a uniform screw travels in a fixed close-fitting nut, the distance the screw advances is proportional to its rotation. For example, if the "pitch" of the screw is 0.5 mm. it advances 0.01 mm. in $\frac{1}{50}$ of a complete rotation. The linear scale is attached to the nut, the smallest unit of the scale being equal to the pitch of the screw. A circular scale attached to the screw makes it possible to divide the smallest unit of the linear scale into as many parts as there are divisions on the circular scale.

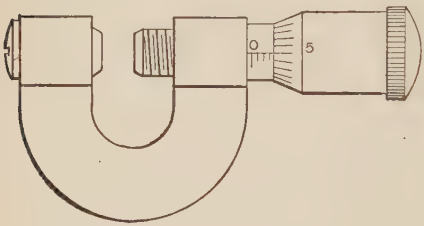

FIG. 3. — Micrometer Screw Gauge.

4. Unit of Time. — The *mean solar second* is $\frac{1}{86400}$ of the mean solar day, which is the average, throughout a year, of the intervals that elapse between successive transits of the sun across the meridian. The mean solar minute equals 60 mean solar seconds, and the mean solar hour equals 60 mean solar minutes.

5. Units of Angle. — The *radian* is the angle at the centre of a circle of radius r subtended by an arc of length r. An angle at the centre of a circle of radius r

subtended by an arc of length a contains $\frac{a}{r}$ radians. Since the circumference of a circle is 2π times the radius, 4 right angles equal 2π radians and 1 right angle equals $\frac{1}{2}\pi$ radians.

The *degree* is the ninetieth part of a right angle. Since $360°$ equals 2π radians, 1 degree equals $\frac{2\pi}{360}$ radians and 1 radian equals $57°.29578$.

Exercise I. The Principle of the Vernier

To construct a direct vernier to accompany a scale, a length equal to $n-1$ of the smallest units of the scale must be divided into n parts on the vernier. For instance, to supply an inch scale divided to tenths of an inch with a vernier reading to hundredths of an inch, take a strip of paper about 2 in. long and lay off on it very carefully a length equal to $\frac{9}{10}$ of an inch. This length must next be divided into 10 parts. This may be done with the aid of a piece of cross-section paper, provided the smallest division of the paper be less than that of the scale.

A direction is to be found on the cross-section paper such that the distance in that direction between two parallel lines separated by 10 spaces equals $\frac{9}{10}$ of an inch. The vernier strip being placed in that direction, the two lines mentioned and the intervening ones will subdivide it into 10 parts. Slight dents corresponding to the points of subdivision may be marked on the vernier strip by means of a sharp knife. Dividing lines should then be drawn through the dents with a sharp pencil and a small square. The vernier should then be fastened by thumb-tacks on a strip of wood and the vernier divisions numbered in the proper direction. The accuracy with which the vernier has been subdivided may be tested by slipping it along the scale and noticing whether the difference between each scale division and each vernier division seems fairly constant.

A "retrograde" vernier should be constructed in a similar way by dividing $\frac{11}{10}$ of an inch into 10 parts on the vernier. It may be

mounted on the other side of the same strip of wood, and its divisions should be numbered in the proper direction.

With each of these verniers and the inch scale, two measurements should be made of each of the dimensions of several small regular blocks. Each measurement will consist in finding, (1) the zero reading, that is, the point on the scale opposite the zero of the vernier when the end of the scale and the end of the wooden vernier strip coincide, and (2) the position of the zero of the vernier when the scale is held vertically on a smooth plane surface and the object is placed in position under the vernier.

A blank form for tabulating these measurements should be devised and drawn neatly. Every separate measurement should be recorded.

For practice in the use of the micrometer caliper, some of the blocks should also be measured by that instrument.

DISCUSSION

(*a*) Comparative merits of "direct" and "retrograde" verniers.

(*b*) "Least count" of a vernier; how it depends on the number of divisions of the vernier.

(*c*) Reading of barometer vernier and other verniers.

(*d*) Comparative merits of the vernier and the micrometer screw method of subdivision.

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Gray's "Treatise on Physics," Vol. I, §§ 8–14, on the "Measurement of Time."

Stewart and Gee's "Elementary Practical Physics," Part I, Chapters I and II, on "Measurement of Length" and "Angular Measurements."

Encyclopædia Britannica, 10th edition, "Weights and Measures."

CHAPTER II

POSITION AND DISPLACEMENT

6. Position. — The position of a point cannot be stated definitely without reference to some other point which may be called the starting point. The statement of a position is in reality the description of a path leading from the starting point to the position described. In other words, "all position is relative."

Universal experience shows that a complete statement of the position of a point in space always requires the use of at least *three numbers*. For instance, to get from a starting point at sea level to the top of a mountain or the bottom of a mine one must go a *certain distance* east or west, a *certain distance* north or south and a *certain distance* up or down. Or one might go a *certain distance* in a direction that makes a *certain angle* with the north and south line and a *certain angle* with the horizontal plane through the starting point. In other words, *space is of three dimensions.*

7. Rectangular Coördinates. — The three numbers that are necessary in order to state completely the position of a point are called the *coördinates* of the point. When the numbers are lengths in some three directions at right angles, the coördinates are called *rectangular coördinates*, and lines in these three directions intersecting at the starting point are called the *axes* of coördinates. The

starting point is called the *origin* of coördinates. Three rectangular coördinates are usually denoted by x, y, and z, and the corresponding axes are called the x-axis, the y-axis, and the z-axis respectively. One direction along an axis is called the *positive* direction of the axis and the opposite the *negative* direction, and values of x, y, and z in these directions are marked by the signs $+$ and $-$ respectively. Any reader who is not quite familiar with these ideas should consider the coördinates of various points in a rectangular room of length l, breadth b, and height h, the origin being taken at a corner, the centre of one side, and the centre of the room successively.

When all the points considered in any case are known to lie in a single plane, two rectangular axes and two coördinates are sufficient. In this case the third item of information is that which fixes the position of the plane.

8. Displacement. — A displacement is a change of position. A displacement evidently cannot be clearly specified without a statement of its *direction* as well as of its *magnitude*. If a point starts from A and arrives at B, the magnitude of the displacement is the length of the straight line AB and the direction of the displacement is the direction of the line AB drawn from A to B. The symbol $\overrightarrow{AB}$ or $\overline{AB}$ may be used as an abbreviation of the phrase "the displacement whose length is AB and whose direction is from A to B."

A point that starts from A and arrives at B may have moved along the straight line AB or it may have taken any irregular path such as $ACDB$. Hence a displacement $\overline{AB}$ may be regarded as the sum of a series of successive

displacements if the starting point of the series be at A and the ending point at B or

$$\overline{AC}+\overline{CD}+\overline{DB}=\overline{AB}.$$

This equation may be read "a displacement $\overline{AC}$ *followed by* a displacement $\overline{CD}$ *followed by* a displacement $\overline{DB}$ *is equivalent* to the displacement $\overline{AB}$." Thus the sign of addition does not mean the addition of mere numbers or of quantities that may be represented by lengths along the same line, as in ordinary Algebra; nor does the sign of equality mean equality of mere numbers or of lengths. The addition of displacements is a *geometrical* addition.

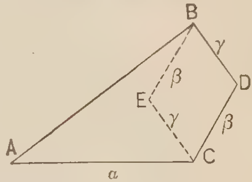

FIG. 4.

A zero displacement is one which leaves the position of the point unchanged. Since $\overline{AB}+\overline{BA}=\overline{AA}=0$, it follows that $\overline{BA}=-\overline{AB}$. Thus subtraction of a displacement is the same as the addition of an equal and opposite displacement.

The displacements we shall be concerned with are independent displacements, the occurrence of any one does not in any way interfere with the occurrence of any other. Hence they may be supposed to take place in any order, and a consideration of Fig. 4 will make it clear that the result of the addition is independent of the order of addition. Thus the addition of three displacements a, β, γ in the order a, β, γ is represented by the figure $ACDB$, and their addition in the order a, γ, β is represented by the figure $ACEB$; the result in both cases is the displacement $\overline{AB}$. This would not be so unless a displacement

represented by BD were equally well represented by an equal and parallel line CE taken in the same direction. *All displacements are considered as equal which have the same magnitude and the same direction.*

9. Geometrical Methods of adding Displacements. — The following propositions summarize the preceding and are convenient for future reference.

1. The *Triangle Method.* The sum of two displacements $\bar{AB}$ and $\bar{BC}$, where AB and BC are two sides of a triangle ABC, is $\bar{AC}$.

2. The *Parallelogram Method.* The sum of two displacements $\bar{AB}$ and $\bar{AC}$, where AB and AC are two sides of a parallelogram $ABDC$, is $\bar{AD}$.

3. The *Polygon Method.* The sum of any number of displacements $\bar{AB}$, $\bar{BC}$, $\bar{CD} \cdots \bar{NP}$, where AB, BC, $CD \cdots NP$ are sides of a polygon, is $\bar{AP}$.

10. Translation. — A change of position of a body is called a *translation* when all points in the body move equal distances in parallel lines, *i.e.* when they undergo equal displacements. When the displacements are not equal the body is in rotation or undergoes *angular displacement* about some axis. We shall postpone the consideration of angular displacements for the present.

11. Addition of Simultaneous Displacements. — A ball rolled a certain distance across the deck of a ship while the ship moves a certain distance forward undergoes, relatively to the earth, two simultaneous displacements, one forward, one sideward. Moreover the displacements are

quite independent, one does not interfere with or influence the other; the ball would move the same distance sideward if the ship did not move, and it would move the same distance forward if it were not rolled sideward. The position of the ball at the end of a second is the same as if it were first moved forward the distance the ship moves in a second, and then moved sideward the distance it rolls in a second. Hence it is obvious that simultaneous independent displacements may be added as if they were successive displacements.

12. Formulæ for Resultant of Two Displacements. — The sum of any number of displacements is also called the *resultant* of the displacements and the displacements are said to be *components* of the resultant.

The magnitude of the resultant of two displacements can readily be found by trigonometry. Let the displacements be $\overline{AB}$ and $\overline{AC}$ and their resultant $\overline{AD}$. Let the magnitude of AB be d_1, of AC d_2, and of AD d.

Then from the triangle ABD

$$d^2 = d_1^2 + d_2^2 - 2\,d_1 d_2 \cos ABD.$$

If θ be the angle between the positive directions of $\overline{AB}$ and $\overline{AC}$, then $\angle ABD = \pi - \theta$ and therefore

$$d^2 = d_1^2 + d_2^2 + 2\,d_1 d_2 \cos \theta.$$

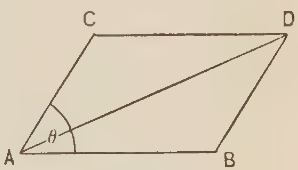

FIG. 5.

If $\overline{AB}$ and $\overline{AC}$ be at right angles and their resultant make an angle θ with AB, then

$$d^2 = d_1^2 + d_2^2 \text{ and } \tan \theta = \frac{d_2}{d_1}.$$

13. Resultant of Three Rectangular Displacements. — The resultant of three rectangular displacements $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, is $\overline{OD}$, where OD is the diagonal of a rectangular parallelopiped of which OA, OB, OC, are intersecting edges.

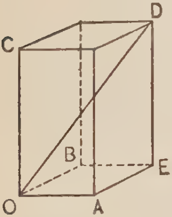

FIG. 6.

For $\overline{OB} = \overline{AE}$ and $\overline{OC} = \overline{ED}$

and $\overline{OD} = \overline{OA} + \overline{AE} + \overline{ED}$

$$= \overline{OA} + \overline{OB} + \overline{OC}.$$

If the magnitudes of $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, be d_1, d_2, d_3 respectively, and the magnitude of the resultant d, then

$$d^2 = d_1^2 + d_2^2 + d_3^2.$$

14. Resolution of a Displacement into Components.— When a displacement is replaced by components in given directions, it is said to be "resolved into components in those directions," or, briefly, "resolved in those directions." A displacement can be resolved into components in any number of specified directions, provided a polygon can be drawn of which one side represents the displacement and the other sides are in the directions specified.

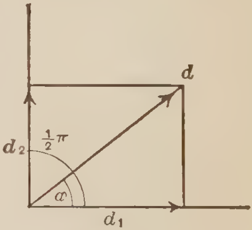

FIG. 7.

If a displacement d be resolved into a component d_1 in a direction making an angle α with d, and a component d_2 in a direction making a right angle with the first component d_1 then

$$d_1 = d\cos\alpha,\ d_2 = d\sin\alpha, \text{ and } d^2 = d_1^2 + d_2^2.$$

In stating these equations, it is understood that the angle α and the right angle that the direction of d_2 makes with the direction of d_1 are both measured in the same direction (say counter-clockwise) from the direction of d_1.

Similarly, a displacement d may be resolved into components in three directions at right angles, namely, d_1, d_2, d_3, and, as is obvious from Fig. 6, $d^2 = d_1^2 + d_2^2 + d_3^2$.

15. Analytical Method of adding Displacements. — Consider any number of component displacements, $d_1, d_2, d_3, \cdots$, in the same plane, and let the angles they make with the positive direction, OX, of a line in that plane be $\alpha_1, \alpha_2, \alpha_3, \cdots$ respectively, the angles being all measured in the same direction (*e.g.* counter-clockwise) from the positive direction of OX.

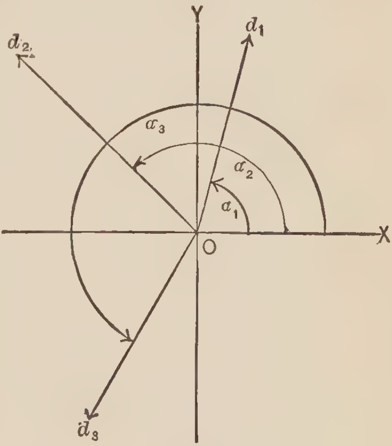

FIG. 8.

Let each of the displacements be resolved into a component in the direction of OX and a component in a direction OY, making a right angle with OX. If the sum of the components in the direction of OX be denoted by D_x, and the sum of the components along OY by D_y, then

$$D_x = d_1 \cos \alpha_1 + d_2 \cos \alpha_2 + d_3 \cos \alpha_3 + \cdots = \Sigma d \cos \alpha.$$

$$D_y = d_1 \sin \alpha_1 + d_2 \sin \alpha_2 + d_3 \sin \alpha_3 + \cdots = \Sigma d \sin \alpha.$$

The resultant of D_x along OX and D_y along OY is a displacement D making an angle θ with Ox, and by § 12,

$$D^2 = D_x^2 + D_y^2 \text{ and } \tan\theta = \frac{D_y}{D_x}.$$

When $\tan\theta$ is positive, the angle θ will be between 0 and 90° if D_x and D_y be both positive, and between 180° and 270° if D_x and D_y be both negative. When $\tan\theta$ is negative, θ will be between 90° and 180° if D_x be negative and D_y positive, and between 270° and 360° if D_x be positive and D_y negative.

If $D_x = 0$ and $D_y = 0$, then $D = 0$. The converse is also true, for D_x^2 and D_y^2 cannot be negative: hence, if $D = 0$, then $D_x = D_y = 0$.

If the component displacements be not in the same plane, they may be resolved into components in three directions, OX, OY, OZ, at right angles. If the sums of the components in these directions be D_x, D_y, D_z respectively, then the final resultant D is such that $D^2 = D_x^2 + D_y^2 + D_z^2$. As before, if D_x, D_y, and D_z be all zero, D will also be zero, and conversely.

Exercise II. Addition of Displacements

Find (1) graphically by the polygon method, (2) by the analytical method, the sum of the following displacements in the same plane, the angle between each and a fixed direction, say from west to east, being as indicated in brackets:

8 (0°), 10 (60°), 6 (115°), 4 (150°), 7 (190°), 8 (270°), 12 (350°).

The magnitudes and directions of the displacements should be very carefully laid off by a protractor and scale. The addition should be repeated, the components being taken in the reverse order and the origin being the same.

In applying the second method, the components of the displacements should be tabulated, all the x-components being in one vertical column and all the y-components in another.

DISCUSSION

(a) Sources of error in applying graphical method.

(b) Can a displacement be resolved into components in *any* two assigned directions? In any three?

(c) Which of two component displacements more nearly coincides in direction with the resultant?

(d) Show how to resolve a displacement into two others of given magnitudes. When is this impossible?

(e) What is the resultant of two displacements, $\overline{m \cdot OA}$ and $\overline{n \cdot OB}$?

(f) The component, in any direction, of the resultant of two displacements equals the sum of the components of the two displacements in that direction.

16. Vector and Scalar Quantities. — Anything that can be measured in terms of a unit of the same kind is called a *quantity*.

Quantities may be divided into two classes. Those which have magnitude but not direction are called *scalar* quantities, because they are measured merely in terms of certain scales or units. Mass, volume, density, etc., are scalar quantities. Quantities which have direction as well as magnitude are called *vector* * quantities, because the simplest vector quantity, a displacement, may be regarded as a *carrying* of a body from one point to another. Other examples of vector quantities are velocity, acceleration, force, etc.

* Derived from the root of the Latin verb for *carry;* compare con*vect*ion.

17. Graphical Representation of Vector Quantities. — A diagram in geometry is a graphical representation of the distances and directions of things in space. The lines in an accurate diagram are (either really or perspectively) proportional in length to the distances they represent, and the angle between any two lines is equal to the angle between the directions they represent. An accurate diagram will remain accurate if all its dimensions be changed in a constant ratio, or if it be turned around into any new position.

Precisely similar statements are true of any diagram of displacements such as we have already employed. The statement that a certain line represents a certain displacement means that the line is one in a diagram of lines representing displacements, that the ratio its length bears to that of any other line is the ratio of the magnitudes of the displacements they represent, and that the angle between the two lines equals the angle between the actual displacements. A line in such a diagram is called a *vector*. Thus a vector is a line which has a definite length and a definite direction relatively to other similar lines in a diagram.

Any other vector quantity, *e.g.* a velocity, is conveniently represented in the same way by means of a line in a diagram. Such a diagram can represent only vector quantities of the same kind, *i.e.* if one line represents a velocity, all lines in the diagram represent velocities. For brevity we may indicate any vector quantity, represented in a diagram, by the symbol already used for representing a displacement. For example, "the velocity $\overline{AB}$" means the velocity represented in a diagram of velocities by the line AB.

It is not obvious that, because other vector quantities may be represented in the same way as displacements, they may be added by the same methods. This, however, is true of the vector quantities we shall be concerned with, but before assuming it we shall prove that the addition may be reduced to an addition of displacements.

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Clifford's "The Common Sense of the Exact Sciences," Chapter IV, "Position," §§ 1–4.

Maxwell's "Matter and Motion," Chapter I.

Clifford's "Elements of Dynamic," Chapter I, on "Steps."

Love's "Theoretical Mechanics," Chapter II, "Geometry of Vectors."

CHAPTER III

VELOCITY AND ACCELERATION

18. Velocity. — The rate of displacement of a point is called its *velocity*. Hence, velocity is a quantity that has both magnitude and direction, that is, it is a vector quantity. When the displacements in equal intervals, however short, are equal in both magnitude and direction, the velocity is a *constant* or *uniform velocity*. When this condition is not satisfied, the velocity is variable.

The measure of a constant velocity is the displacement it produces in unit time if it remains constant that long. Otherwise it is the displacement in the fraction of a unit, during which the velocity is constant, multiplied by the number of such fractions in unit time.

Rate of motion without reference to the direction of the motion is called *speed*. Two ships have the same speed if each travels 10 mi. per hour, but they have not the same velocity unless they move in the same direction.

Velocities and speeds, like displacements, are essentially relative, that is, we cannot specify the velocity or speed of a point without reference to some other point So, too, rest is only a relative term.

19. Composition and Resolution of Constant Velocities. — The resultant of two constant velocities is the single velocity that would produce in a certain time a displacement equal to the sum of the displacements produced by the two

velocities. A bird flying northward in a current of air that has an equal velocity eastward has two component velocities, but an observer at a distance would only be aware of the fact that the bird is moving northeast.

The measure of the resultant of two velocities is the resultant displacement in unit time, that is, the sum of the displacements produced in unit time by the two velocities. Hence the various methods (triangle, parallelogram, polygon, and analytical method) that may be used for adding displacements may also be used for finding the resultant of velocities. Conversely, velocities may be resolved into components as displacements are resolved.

20. Variable Velocity. — The *mean velocity* of a point in any interval of time is its displacement in that interval divided by the interval. This definition will apply also to a point whose velocity is constant; but the mean velocity will equal the constant velocity, since either multiplied by the interval will give the displacement.

When the velocity of a point is variable, the velocity of the point at any instant, or its *instantaneous velocity*, *is the value approached by the mean velocity in an interval including that instant, if the interval is taken shorter and shorter without limit.* If Δs be the displacement in an interval Δt, $v = \text{limit}\ \dfrac{\Delta s}{\Delta t}$, as Δt approaches zero. Consider, for example, the mean velocity of a train in 10 sec., 0.1 sec., 0.001 sec., and so on. The smaller the interval the more the velocity approaches a definite value, namely, the instantaneous velocity at any instant in the interval.

The instantaneous velocity, as defined above, has a definite magnitude and direction at any instant. A constant velocity of the same magnitude and direction would be measured by the displacement it would produce in unit time. Hence the instantaneous velocity of a point equals the displacement produced in unit time by an equal constant velocity.

To further illustrate the meaning of an instantaneous velocity consider the following: A train, A, whose velocity is increasing, is passing a train, B, moving with a constant velocity in the same direction; a passenger on A observes that the velocity of B seems first to decrease, then to become zero, then to reverse. At the instant of relative rest, the *instantaneous* velocity of A equals the constant velocity of B. If we divide the displacements of A and B in a short interval, including the instant of relative rest, by the length of the interval, and suppose the interval indefinitely short, we will get the instantaneous velocity of A and the equal constant velocity of B.

21. Composition of Instantaneous Velocities. — An instantaneous velocity may be measured by the displacement it would produce in unit time if it continued constant that long; similarly for a second instantaneous velocity. These supposed displacements may be compounded by the triangle, parallelogram, polygon, or analytical method. Hence instantaneous velocities may be similarly compounded.

22. Acceleration. — A change of velocity may be an increase of speed, as in the case of a train leaving a station, a decrease of speed, as in the case of a train approaching a station, or a change in the direction of the velocity,

as in the case of a train rounding a curve, or it may be a change of direction accompanied by a change of speed, as in the case of a train approaching or leaving a station on a curve. Any such change of velocity is called an *increment of velocity.* An increment of a velocity has a definite magnitude and a definite direction, or it is a vector quantity.

The rate of change of a velocity is called the *acceleration* of the velocity. When the increments of velocity in all equal times are equal as regards both magnitude and direction, the acceleration is called a *constant acceleration.* A constant acceleration may be measured by the increment of velocity to which it gives rise in unit time. Hence, like increment of velocity and displacement, acceleration is a vector quantity.

When an acceleration is variable the *mean acceleration* and the *instantaneous acceleration* are defined as in the analogous case of variable velocity. The reader should construct these definitions for himself.

23. Composition of Accelerations. — Since accelerations are measured by the increments of velocity to which they give rise in unit time, accelerations may be compounded and resolved as velocities and displacements are compounded and resolved.

24. Constant Acceleration in the Line of Motion. — The simplest case of a constant acceleration is when the increments of velocity are in the direction of the velocity, that is, when they are increases or decreases of speed. This is the case of a body projected vertically upward or downward or a train leaving or approaching a station on a

straight horizontal track (though in the latter case the acceleration may sometimes be variable in magnitude).

If the constant acceleration be a and the velocity at the beginning of a certain interval of time be u, then the increment of velocity in the time t is at and the velocity at the end of the time is

$$v = u + at. \tag{1}$$

Since the velocity increases uniformly, its average value in the interval is equal to the velocity at the middle of the interval or $u + \frac{1}{2}at$. Hence the distance traversed in time t is

$$s = (u + \tfrac{1}{2}at)t = ut + \tfrac{1}{2}at^2. \tag{2}$$

(A more satisfactory proof of this equation is suggested by (*c*) in "discussion" of Exercise IV.)

Eliminating t from (1) and (2), we get

$$v^2 = u^2 + 2as. \tag{3}$$

25. Acceleration of Gravity. — It has been found by experimental methods that the speed of a body falling freely in a vertical line increases by about 32.2 ft. per second in every second, or 980 cm. per second in every second. (These are only approximate figures, since the increase is slightly different at different places; see Table in Appendix.) If a body is moving vertically upward, its speed decreases at the rate stated. If the body's motion is at some inclination to the vertical, the component of its velocity vertically downward increases at the above rate. Briefly stated, the acceleration of gravity is vertically downward, and its magnitude is 32.2 ft. per second per second, or 980 cm. per second per second.

26. Units and Dimensions. — The *fundamental* units used in kinematics are the unit of length and the unit of time. Other units defined in terms of these are called *derived* units. A system of units in which the derived units bear the simplest possible relation to the fundamental units is called an *absolute system*. In such a system the unit of surface is the square of the unit of length, or, as it may be briefly expressed, $(S) = (L)^2$. Similarly, the unit of volume $= (L)^3$. The unit of velocity in an absolute system is unit length per unit time, and its magnitude therefore varies directly as the magnitude of the unit of length, and inversely as the magnitude of the unit of time, or, briefly, $(V) \propto (L)(T)^{-1}$. The unit of acceleration is unit velocity gained in unit time, and its magnitude therefore varies directly as the magnitude of the unit of velocity, and inversely as the magnitude of the unit of time, or $(A) \propto (V)(T)^{-1} \propto (L)(T)^{-2}$. These relations are called *dimensional* relations. They may be expressed in words thus: velocity is of 1 dimension in length and -1 dimension in time; acceleration is of 1 dimension in length and -2 dimensions in time.

The *numerical measure* of any quantity varies inversely as the magnitude of the unit of the same kind in which it is measured. Thus, a length that is 3 in feet is 1 in yards, and a velocity that is 3 in feet per second is 1 in yards per second. These are simple cases of changes of units; more complicated cases are most readily worked out by means of dimensional relations. For instance, suppose an acceleration of 32.2 ft. per second per second is to be expressed in metres per minute per minute. The dimensional rela-

tion of acceleration to length and time is $(A) \propto (L)(T)^{-2}$, and the numerical measure of the acceleration varies inversely as the unit of acceleration. Denoting the measure of the acceleration in metres per minute per minute by x,

$$\frac{x}{32.2} = \frac{(\text{foot})(\text{sec.})^{-2}}{(\text{metre})(\text{min.})^{-2}} = \frac{(\text{foot})}{(\text{metre})} \cdot \frac{(\text{min.})^2}{(\text{sec.})^2} = \frac{30.48^*}{100} \cdot \left(\frac{60}{1}\right)^2.$$

$$x = 35.3 \times 10^4.$$

(The student should note in the statement of the result the use of powers of 10 to express large numbers and also that, since 32.2 is only given to one-third of one per cent, it is not worth while calculating x to a higher degree of apparent accuracy.)

27. Motion of a Point that has a Constant Velocity in one direction and a Constant Acceleration in a direction at right angles. — A consideration of any such case will make it clear that the two parts of the motion may be considered as taking place separately and independently. Thus, a ball thrown upward in a train moving uniformly returns to the hand. Relatively to the earth the ball describes a curve, but throughout its motion it keeps its horizontal velocity unchanged, and its vertical motion is the same as if it had no horizontal velocity. When the velocity of the train is changing rapidly, the ball does not return to the hand. The reader should think this illustration out carefully, assuming different values for the acceleration of the train and the time the ball is in the air.

* 1 ft. = 30.48 cm. approximately.

When a body is projected from the earth in any direction, its horizontal displacement, x, at any time, t, and the vertical displacement, y, at the same time may be calculated separately. If v_1 be the horizontal component of the velocity of projection, and v_2 the component vertically upward, then, since the acceleration of gravity, g, is vertically downward,

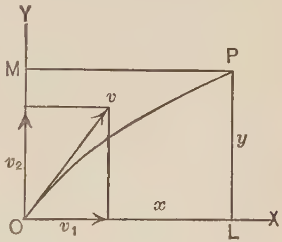

Fig. 9.

$$x = v_1 t,$$

$$y = v_2 t - \tfrac{1}{2} g t^2.$$

From these two equations we may eliminate t and get an equation connecting x and y that holds true for any simultaneous values of x and y. This equation is called the equation of the curve which the body describes. If the body be projected horizontally from an elevation, then $v_2 = 0$ and v_1 equals the velocity of projection. If the positive direction of y be taken downward, $x = v_1 t$ and $y = \frac{1}{2} g t^2$. In this case the constant relation between x and y may be written

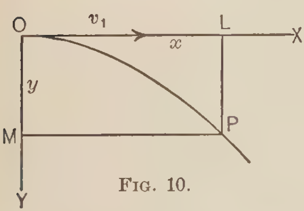

Fig. 10.

$$\frac{x^2}{y} = \frac{2 v_1^2}{g} \text{ (which is a constant).}$$

The horizontal velocity at any time is v_1, and the vertical velocity $v_2 = gt$. If V be the resultant velo-

city at time t, and θ the angle it makes with the horizontal,

$$V^2 = v_1{}^2 + v_2{}^2 \text{ and } \tan\theta = \frac{v_2}{v_1}.$$

Exercise III. Path of a Projectile

Apparatus. — A cross-sectioned board* is mounted in a vertical plane with its lines horizontal and vertical respectively. At one of the upper corners of the board a block in which there is a curved groove is attached to the board in such a way that it is adjustable in a vertical plane. A steel ball rolling down the groove is projected in a horizontal direction, and after describing a curved path in front of the board is caught in a small bag or pocket. A simple spring release worked by a cord enables the observer to drop the ball while he is standing in front of the board in a position convenient for observing the falling ball.

Observation of Curve of Descent. — The apparatus should first be adjusted by means of a level so that one set of lines is truly horizontal and the lower straight end of the "shoot" is also horizontal. These adjustments should be occasionally retested. It is necessary that in successive observations the release should be worked as uniformly as possible. For this purpose a small weight is attached to the end of the cord; when the ball is to be dropped the small weight is pushed off a platform and, falling three or four centimetres, jerks the cord and releases the ball. Another simple method that sometimes gives more satisfactory results is to remove the platform that supports the weight and raise the weight, by means of the cord, to some definite position and allow it to fall three or four centimetres. Care must, however, be taken that the fall is not so great as to alter the spring that holds the ball.

The intersection of the curve of descent with each horizontal line of the board is to be observed about six times in as many successive

* The ruled board referred to in this and subsequent exercises may with some advantage be replaced by a plain board on which a sheet of cross-section paper is fastened by thumb-tacks. The board will probably shrink somewhat, and the shrinkage will be different across and along the grain.

falls and the mean taken. Each reading should be estimated to one-tenth of the smallest division of the horizontal line. The observer

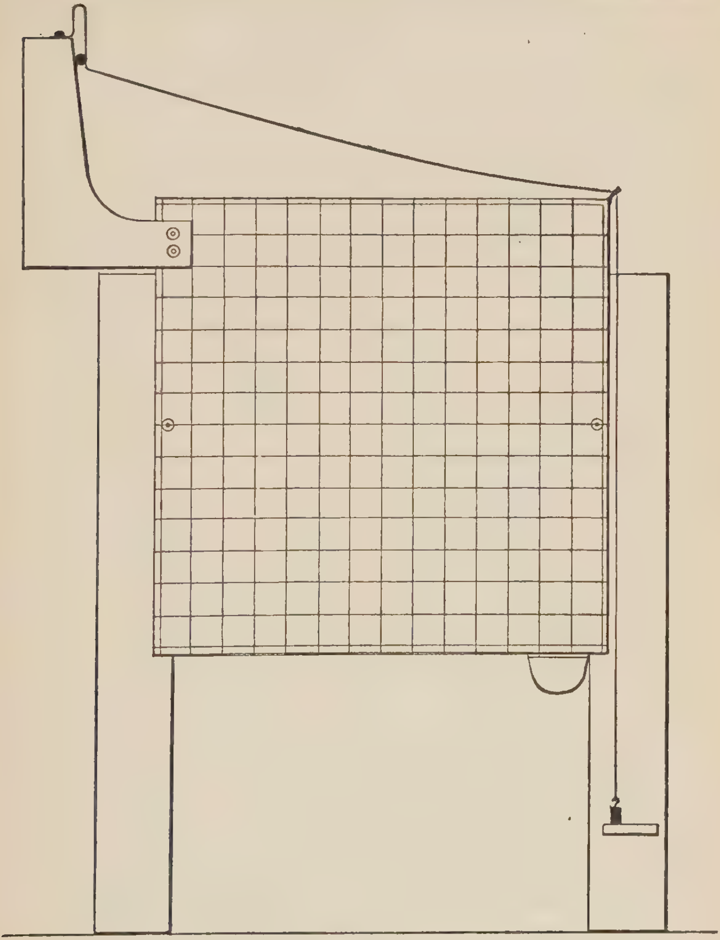

FIG. 11.

should endeavor to avoid "parallax" by holding his head in such a position at the moment of observation that the gaze is fixed at right

angles to the board. The lines may be taken in any order found convenient. There is some advantage in taking the first, third, fifth, etc., lines and afterward the second, fourth, etc. In this way a complete curve may be obtained even if time does not permit all the lines to be observed. Moreover, the observer becomes more expert with practice and the increased accuracy will be more evenly distributed.

All these readings should be arranged in tabular form, the readings referring to any one horizontal line being in a vertical column. When the readings have been completed, the mean values should be plotted on cross-section paper, the origin being taken at an upper corner of the paper in imitation of the position of the board. A smooth curve should then be drawn, striking an average path among the points located but not necessarily passing through any particular point. For drawing the curve celluloid curve-forms should be used.

Calculation of Initial Velocity. — The constancy of $\frac{x^2}{y}$ may be tested by means of the values obtained for each horizontal distance and the corresponding vertical distance. Before this is done care should be taken to ascertain whether the ball at the moment of discharge from the "shoot" was exactly at the origin of the cross lines of the board. If not, allowance for the initial position of the ball must be made by subtracting its initial vertical and horizontal distances from the mean values recorded for the other points on the curve. Moreover, if the unit of the board is not a centimetre, the corrected horizontal and vertical distances should be reduced to centimetres before the calculations of $\frac{x^2}{y}$ are made.

The initial velocity of the ball is calculated from the final mean value of $\frac{x^2}{y}$.

DISCUSSION

(*a*) Meaning and derivation of formulæ.

(*b*) Consider the motion of a line passing through the ball in the experiment and through another ball discharged simultaneously with the same velocity in the same direction but supposed devoid of weight.

(*c*) Consider the motion of a line passing through the ball and through another ball allowed to fall simultaneously from the same point.

(*d*) At what point on the curve and at what time was the direction of motion of the ball inclined at 45° to the horizontal? At 30°? At 60°?

(*e*) Show that V^2 increases as if the fall were wholly vertical.

(*f*) From what height would the ball have to fall freely to attain its initial velocity?

(*g*) Could the initial velocity be deduced from the height of the groove?

(*h*) Does the rotation of the ball affect the results?

(*i*) Equations of the vertical and horizontal motion and of the path when a ball is discharged obliquely.

(*j*) What initial velocity must a bullet have to fall back to its starting point in 10 sec.?

(*k*) A body is projected at an angle of 30° with the horizontal with a velocity of 30 m. per second. When and where will it again meet the horizontal plane through the starting point? How high will it ascend? (First find how long the vertical motion lasts.)

(*l*) What is the final speed of a body which, moving with uniform acceleration, travels 72 m. in 2 min. if:

(1) the initial speed = 0?
(2) the initial speed = 15 cm. per second?

(*m*) At what angle with the shore must a boat be directed in order to reach a point on the other shore directly opposite if the speed with which the boat is rowed be 4 mi. an hour and that of the stream 3 mi. an hour?

(*n*) A raindrop, falling nearly vertically in still air, makes a "streak" inclined at 30° to the vertical on the pane of a railway car travelling at 25 mi. an hour. What is the velocity of the drop?

28. Curve of Speed. — A *curve of speed* is a convenient method of showing graphically the way in which the speed of a body varies. A horizontal line OT is drawn to represent time reckoned from some moment represented by O. At the end T_1 of a length OT_1 that represents an interval t_1 an ordinate T_1S_1 is erected to represent the

speed at the end of the interval t_1. Similar ordinates are erected at points $T_2, T_3 \ldots T_n$. A smooth curve through $S_1, S_2, S_3 \cdots S_n$ is the curve of speed of the moving point. When the curve has been drawn the speed at the end of any interval t can be found by measuring the ordinate at the corresponding point T. (TS is said to be found by *interpolation.*) This diagram has the property that the distance traversed by the body in any interval T_1T_n is represented by the area bounded by the curve, the horizontal time line, and the ordinates at T_1 and T_n. This is evident from the fact that the distance traversed in any very short time, τ, is equal to the speed at the middle of the time multiplied by τ and is, therefore, represented by the area of the narrow trapezium that stands on the length representing τ.

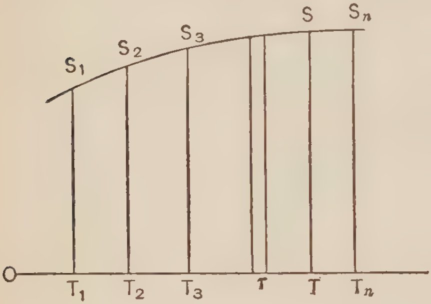

FIG. 12.

29. Uniform Circular Motion. — When a point revolves in a circle at a constant rate, the magnitude of its velocity, that is, its speed, remains constant, but the direction of the velocity is constantly changing.

If s denote the speed, then s is the length of the arc described in unit time. The angle through which the radius turns

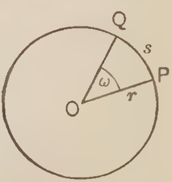

FIG. 13.

in unit time is $s \div r$ radians. This is called the *angular velocity* and is denoted by ω. Hence $\omega = \frac{s}{r}$.

If T is the time required for a revolution, since the radius turns through 2π radians in time T

$$\omega = \frac{2\pi}{T}.$$

If n is the *frequency* of the motion or the number of revolutions in unit time, $nT = 1$ and

$$\omega = 2\pi n.$$

30. Accelerated Angular Velocity. — When an angular velocity increases at a constant rate, it is said to be subject to a constant angular acceleration. If we denote the constant angular acceleration by a we may say that a is the change of ω in time t divided by t. If ω_0 is the angular velocity at the beginning of t and ω that at the end of t

$$\omega = \omega_0 + at.$$

The mean angular velocity in this time t is the angular velocity at time $\frac{1}{2}t$ or $\omega_0 + \frac{1}{2}at$. Hence the angle described in time t is

$$\begin{aligned}\phi &= (\omega_0 + \tfrac{1}{2}at)t \\ &= \omega_0 t + \tfrac{1}{2}at^2.\end{aligned}$$

These formulæ are analogous to the formulæ for accelerated linear velocity, angular displacement corresponding to linear displacement, angular velocity to linear velocity, and angular acceleration to linear acceleration (see (*d*) p. 37).

When a rigid body is in rotation about an axis each point in the body rotates in a circle whose centre is on

that axis and all points necessarily have the same angular velocity and the same angular acceleration. The angular velocity and acceleration of the body are those of any particle in the body.

When a point revolves in a circle of radius r with an angular acceleration, α, its linear speed along the tangent to the circle changes by αr in unit time (§ 29). Hence if a is its linear acceleration $a = \alpha r$.

31. Graphical Representation of Angular Velocities. — When a body rotates about any axis, its angular velocity may be represented in magnitude and direction by a length proportional to the magnitude of the velocity, laid off on the axis of rotation, the direction of this length being related to the direction of rotation as translation to rotation in the motion of an ordinary or "right-handed" screw.

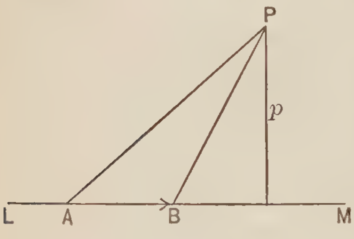

FIG. 14.

Suppose LM to be the axis of rotation and $\overline{AB}$ to be the length laid off on LM to represent the angular velocity. If p is the perpendicular distance of a particle P from LM, the linear speed of P is represented by $p \cdot AB$ (§ 29) or twice the area of the triangle APB, and as $\overline{AB}$* is from left to right in the diagram, the linear speed of P is toward the reader and perpendicular to the plane APB.

* AB is in this case a *localized* vector, since it stands for quantity that has magnitude and direction and also relates to a definite straight line LM, the axis of rotation.

32. Composition of Angular Velocities. — Let us suppose that a body has simultaneously two angular velocities about axes intersecting in A, and let the magnitudes and directions of these angular velocities be represented by $\overline{AB}$ and $\overline{AC}$ respectively. The resultant of these component velocities is a rotation about some axis through A. We shall show that the resultant is an angular velocity about the diagonal AD of the parallelogram $ABDC$ and is represented by $\overline{AD}$. Consider the motion of a point P *in the plane of* $ABDC$. The two separate linear speeds of P are represented by twice the areas of the triangles APB and APC respectively. Now the area of the triangles APD equals the sum of the areas of APB and APC (these triangles are on the same base AP and the distance of D from AP equals the sum of the distance of B and C from AP). Hence twice the area of the triangle APD represents the resultant linear speed of P. Thus P, and therefore every other point in the body, rotates about AD with an angular velocity represented by $\overline{AD}$.

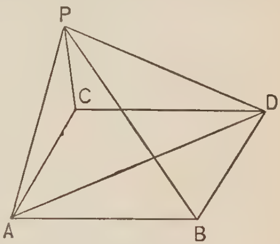

FIG. 15.

Hence, angular velocities about intersecting axes are compounded like linear velocities. The same law evidently holds for angular accelerations, since they are increments of angular velocities in unit time.

The reader should note that the axes we have been speaking of are certain lines in space, not certain lines fixed in the moving body. The difference is important, since a line in the body changes its position as the body

rotates. The propositions hold true, however, for lines in the body that coincide at any moment with the axes in space, provided it be understood that we limit ourselves to the angular velocities at that instant, *i.e.* to instantaneous velocities.

33. Linear Acceleration of a Point that moves in a Circle with Constant Speed. — When a point revolves in a circle with constant linear speed, although the magnitude of the linear velocity is constant, the direction of the linear velocity is constantly changing and hence there is a linear acceleration. When the point is at Q, it is moving in the direction of the tangent QT with a speed s. Let us represent the velocity at that moment by a line oq. Similarly, when the moving point is at Q', its velocity may be represented by oq', and oq and oq' will be equal in length since the speed is constant. From the triangle oqq' we see that the velocity which has been added to oq to produce oq' is qq'. If t is the time in which the moving point passes over the distance QQ', the mean acceleration in the interval t is $qq' \div t$. If Q and Q' be supposed indefinitely close together, the mean acceleration in the interval t will equal the instantaneous acceleration at the middle of t when the moving point was at P, half way between Q and Q'. Hence if a be the acceleration at P, $a = qq' \div t$ and

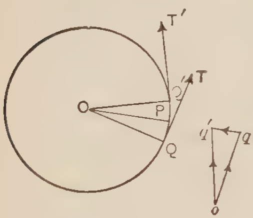

FIG. 16.

$$qq' = at. \tag{1}$$

It is readily seen from geometry that qq' is in the direction PO, or *the acceleration is towards the centre of the circle.*

Since QQ' is traversed in time t with speed s,

$$QQ' = st. \tag{2}$$

From the similarity of the triangles OQQ' and oqq'

$$\frac{qq'}{oq} = \frac{QQ'}{OQ}, \tag{3}$$

and since the arc QQ' is indefinitely short, it may be taken as equal to the chord QQ'. Substituting in (3) from (1) and (2) we get

$$\frac{at}{s} = \frac{st}{r}.$$

Hence
$$a = \frac{s^2}{r},$$

and, as we have seen, a is towards the centre of the circle. The effect of this acceleration is to produce a change in the direction of the velocity without affecting the magnitude of the velocity.

If the angular velocity in the circle is ω and the frequency is n,
$$s = \omega r = 2\,\pi n r.$$
$$\therefore\; a = \omega^2 r = 4\,\pi^2 n^2 r.$$

When a point moves in a circle with varying speed, in addition to the acceleration, a, towards the centre, there must be an acceleration, a', along the tangent. The whole acceleration will be the resultant of these two component accelerations, a and a'.

34. Acceleration of a Point moving in Any Curve. — Through any point P on a curve a circle may be drawn that coincides exactly with the curve at the point P. This circle is called the *circle of curvature* of the curve at P, and its radius r is called the *radius of curvature* at P.

A point moving along the curve on reaching the position P is for a moment moving in the circle of curvature at P and hence has an acceleration $\frac{s^2}{r}$ directed towards the centre of curvature. If the speed of the moving point be variable, there will also be an acceleration along the tangent.

Exercise IV. Curve of Speed of a Projectile

This exercise is a further study of the curve of descent of a falling body obtained in Exercise III.

Let the x or y of any point P on the curve of descent be obtained from the curve; then the time when the ball was at P can be calculated from the equation for the horizontal or that for the vertical motion (§ 27). With a knowledge of t, the component velocities at P can be calculated (§ 24). The resultant velocity can then be found from the components. The values of the velocity as obtained in this way should be tabulated for five or six points on the curve.

With the values of the speed and time draw on cross-section paper a diagram of speed on any convenient scale. The whole distance the ball descended along the curve can be obtained from the diagram of speed by counting up the number of large and small square units in the area of the diagram. This area would equal the length of the curve of descent if each unit of length along the time axis represented a unit of time, and each unit of length along the speed axis represented a unit of speed. But if the former represent m units of time and the latter n units of speed, then each unit of area represents mn units of length of the curve of descent, and the area of the diagram must be multiplied by mn to get the length of the curve of descent.

The length of the curve of descent can also be obtained by direct measurement. Fasten the sheet containing the curve on a board by half a dozen ordinary pins passing through points on the curve, and with a strip of cross-section paper stretched on edge close against the pins, measure the length of the curve. Then allow for the scale to which the curve was drawn.

DISCUSSION AND PROBLEMS

(*a*) Meaning of instantaneous velocity and instantaneous speed.

(*b*) Definition and chief properties of curve of speed (why not "curve of velocity"?).

(*c*) Curve of speed of a body falling in a vertical line. Formula for distance traversed in time t.

(*d*) Curve of angular speed of a body rotating with a constant angular acceleration (§ 30). Formula for angle described in time t.

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Ames's "Text-book of General Physics," Chapter I.
Watson's "Text-book of Physics," Book I, Chapter V.
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Macgregor's "Kinematics and Dynamics," Chapters I–VI.

CHAPTER IV

PERIODIC MOTION

35. Periodic Motion. — When a point repeats a series of movements in successive equal intervals, its motion is called *periodic*. The vibrations of a pendulum, of a mass attached to a spring, of a point which has a uniform circular motion, of a planet rotating about the sun, are periodic motions. The time required for each complete repetition of the motion is called the *period* of the motion. In some respects steady rotation in a circle is the simplest form of periodic motion.

36. Phase and Epoch in Uniform Circular Motion. — When a point P revolves uniformly in a circle, the simplest way of describing its position at any time is by stating the magnitude of the angle, POA, that the radius through P makes with some fixed radius OA, it being understood that POA is measured from OA in some definite direction of rotation, *e.g.* counter-clockwise. The angle POA is called the *phase* of P's motion, and is denoted by ϕ. If E be the position of P at the moment from which time is reckoned, then EOA, or the phase of P's

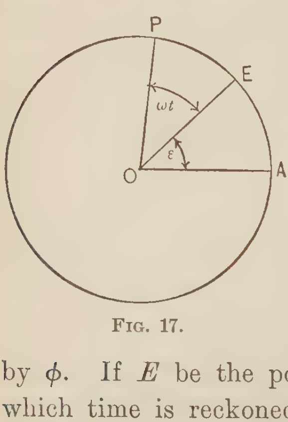

FIG. 17.

motion at zero time, is called the *epoch* of P's motion, and is denoted by ϵ. If P's angular velocity be ω, then at time t $POE = \omega t$, and

$$\phi = \omega t + \epsilon.$$

Phase is sometimes measured by the ratio of POA to 2π, that is, by the fraction of a period that has elapsed since P last passed through the fixed point A. The epoch is then measured by the fraction of a period required by P to move from A to E.

37. Simple Harmonic Motion. — When a point P rotates in a circle with constant speed, the projection of P on a diameter moves backward and forward along the diameter, completing a whole vibration in the time in which P completes a revolution. The motion of M, the projection of P, is called *Simple Harmonic Motion.* Hence, simple harmonic motion is the projection of uniform circular motion on a diameter. The velocity and acceleration of M are the projections of the velocity and acceleration of P.

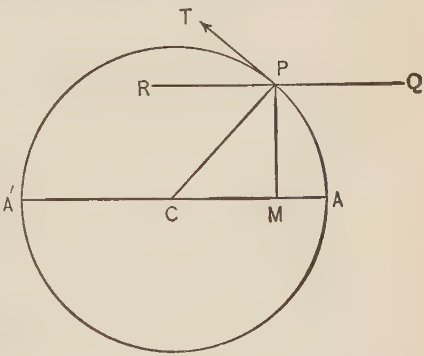

FIG. 18.

It will help the reader to realize the meaning of simple harmonic motion if he imagine himself looking at a uniform circular motion from a very great distance in the plane of the motion. The only part of the motion seen would be the part transverse to the line of sight.

38. Acceleration in Simple Harmonic Motion. — The acceleration of P is $\omega^2 r$ along PC. Let RPQ be parallel to and in the same direction as $A'A$, the diameter on which the motion of P is projected. Then the acceleration of M in the direction CA is $\omega^2 r \cos CPQ$, or $-\omega^2 r \cos PCA$. If CM (or *the displacement* as it is always called in S. H. M.) be denoted by x, $\cos PCA = \frac{x}{r}$. Hence, if a be the acceleration of M in the direction CA,

$$\begin{aligned} a &= -\omega^2 r \cdot \frac{x}{r} \\ &= -\omega^2 x \\ &= -\left(\frac{2\pi}{T}\right)^2 \cdot x. \end{aligned}$$

Hence *the acceleration is opposite to and proportional to the displacement*, and the period of the motion is

$$T = 2\pi\sqrt{-\frac{x}{a}}.$$

39. Velocity in Simple Harmonic Motion. — The velocity of P is ωr along the tangent PT. The velocity of M in the positive direction CA is the component along CA of the velocity of P. Denoting it by v,

$$\begin{aligned} v &= \omega r \cos TPQ \\ &= -\omega r \cos TPR \\ &= -\omega r \sin PCM \\ &= -\omega r \frac{PM}{r} \\ &= -\omega PM \\ &= -\frac{2\pi}{T}\sqrt{r^2 - x^2}. \end{aligned}$$

In this expression, T is the period of the S. H. M., and r equals the greatest displacement in the S. H. M., or the *amplitude* of the S. H. M. The ambiguity of sign due to the square root is removed by considering that v must be positive, as M moves from A' to A, and negative from A to A'.

40. Circle of Reference of a S. H. M. — From § 38 it is seen that S. H. M. is a linear vibration in which the acceleration is opposite to and proportional to the displacement. This might have been, and frequently is, taken as the definition of S. H. M. It was only for convenience and brevity in deducing the properties of S. H. M. that it was defined as projection of uniform circular motion. The reader must guard himself against confusing a S. H. M. and the uniform circular motion from which it may be regarded as projected. The circle described with the centre of a S. H. M. as centre and the amplitude of the S. H. M. as radius is called the *circle of reference* of the S. H. M.

Any linear vibration that has the characteristic that $a = -\mu x$, μ *being a constant throughout the vibration, is a S. H. M., and the period of the vibration is*

$$T = 2\pi\sqrt{-\frac{x}{a}} = 2\pi\sqrt{\frac{1}{\mu}}.$$

Exercise V. Graphical Study of S. H. M.

Method. — A large sheet of paper is tacked to the vertical cross-section board used in Exercise III. We shall suppose that the paper is plain; but the use of accurate cross-section paper, with millimetre divisions, would simplify the work in ways that will be readily seen. On the paper a quadrant PA of a circle is drawn with care by means

of a hard sharp-pointed pencil. (The pencil may be firmly clamped by means of a screw pinch-cock to a rod of wood, through one end of which a sewing needle is driven.) Through the centre of the circle a vertical and a horizontal radius are drawn. The point on the arc through which to draw the vertical radius should be found accurately by means of a plumb-line of fine silk thread, the bob being allowed to hang in a glass of water to prevent oscillations. The direction of the horizontal radius may be found by a metre stick and level. The surface of the paper should be as nearly plane and vertical as possible. The board may be levelled by loosening the screws in the feet of the supports and placing thin wedges under them.

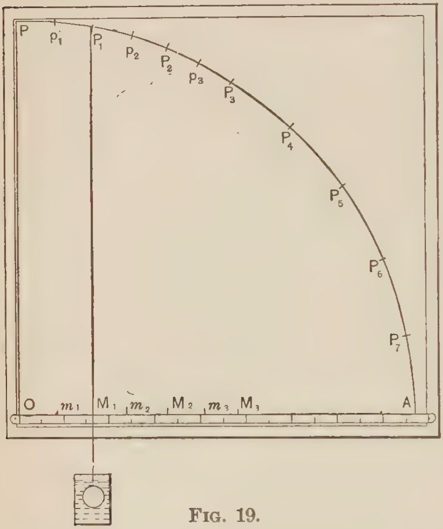

FIG. 19.

Imagine that a point P describes with constant speed the circle of which PA is a quadrant and carries a plumb-line that always remains vertical. The intersection of the plumb-line and OA will have a S. H. M. along OA, its amplitude being OA or r, and its period the same as the period T of P. Divide the arc PA with great care into 8 or 10 equal parts, PP_1, P_1P_2, $\cdots$, and mark the points of division as sharply as possible. P_1, P_2, $\cdots$ will be the position of P after successive equal intervals, each, say, of length τ. If M_1, M_2, $\cdots$ be the projections of P_1, P_2, $\cdots$, they will be the successive positions of M at the ends of these equal intervals and the corresponding displacements of M will be OM_1, OM_2, $\cdots$.

Let p_1 be the middle of the arc PP_1, p_2 that of P_1P_2, and so on,

and let n_1, n_2, $\cdots$ be the projections of p_1, p_2, $\cdots$. Then the velocities of M at the middle of the successive equal time intervals will be proportional to p_1n_1, p_2n_2, $\cdots$ (§ 39).

Measurements. — To find the displacements OM_1, OM_2, $\ldots$, fasten a thin steel tape along OA and note carefully its intersection with the plumb-line estimating to $\frac{1}{5}$ mm. (a metre scale clamped to the framework below the board may be used instead of the tape). The plumb-line should pass accurately through P_1, P_2, $\cdots$. The initial displacement, x_0, of M is 0; $x_1 = OM_1$; $x_2 = OM_2$; $\cdots$. Tabulate these values of x and also the values of p_1n_1, p_2n_2, $\cdots$.

Law of Velocity. — The distance traversed in the successive equal intervals are $d_1 = x_1 - x_0$; $d_2 = x_2 - x_0$; $\cdots$. The mean velocities in these intervals are $v_1 = d_1 \div \tau$, $v_2 = d_2 \div \tau$, $\cdots$. For τ, any value, say $\frac{1}{10}$ sec., may be assumed, and T and ω may then be deduced from the number of parts in the quadrant. The mean velocities in the successive intervals may be taken as the velocities at the middle of the intervals, or $\omega \cdot p_1n_1$, $\omega \cdot p_2n_2$, $\cdots$. Hence $v_1 \div p_1n_1$, $v_2 \div p_2n_2$, $\cdots$ should be nearly equal and their mean should be ω.

Law of Acceleration. — The increment of the mean velocity from the middle of the first interval to the middle of the second interval is $v_2 - v_1$. Hence the mean acceleration in this time is $a_1 = (v_2 - v_1) \div \tau$, and thus may be taken as the acceleration at the end of the first interval, or $-\omega^2 \cdot x_1$. Similarly the acceleration at the end of the second interval is $a_2 = (v_3 - v_2) \div \tau$, and so on. Hence $a_1 \div x_1$, $a_2 \div x_2$, $\cdots$, should be nearly equal and their mean should give $-\omega^2$.

Source of Error. — It will be noticed that we have spoken of *mean* velocities and *mean* accelerations. This is because we could not measure indefinitely short intervals and so get instantaneous velocities and accelerations. It can, however, be shown that if τ is only $\frac{1}{40}$ of T, the errors from this cause are small. Such an error is an *error of method*. There is a more serious source of possible error in this exercise. The first couple of intervals are so nearly equal that, when we take differences to find the acceleration, the difference may be in error by a large fraction of itself, owing to unavoidable errors of measurement, and this may cause the value of a_1 and possibly of a_2 to be very imperfect. Such an error is an *error of measurement* or *of observation*.

DISCUSSION

(*a*) Relation of U. C. M. and S. H. M.

(*b*) Velocity for various values of x, *e.g.* $+r$, $-r$, 0, $+\frac{1}{2}r$, $-\frac{1}{2}r$, etc.

(*c*) Acceleration for various values of x.

(*d*) How, from the velocity or acceleration at a given displacement, could the time required for a whole vibration be calculated?

41. Phase and Epoch in S. H. M.—The term *phase* is used in S. H. M. to denote the position and direction of the motion of the vibrating point at any time. The measure of the phase in the S. H. M. is taken as the same as that in the uniform circular motion from which the S. H. M. may be supposed to be projected. When the point which has S. H. M. is at M (Fig. 18), its phase is AOP. The *epoch* of the S. H. M. is similarly the epoch of the circular motion, so that we have, as in uniform circular motion,

$$\begin{aligned}\phi &= \omega t + \epsilon \\ &= \left(\frac{2\pi}{T}\right)t + \epsilon.\end{aligned}$$

As in uniform circular motion the phase and epoch of a S. H. M. may also be measured in fractions of a period.

42. Resolution of Uniform Circular Motion into Two S. H. M.'s at Right Angles. — If P rotates uniformly in a circle, and if M and N are the projections of P on $A'A$ and $B'B$, two diameters at right angles, then the motions of M and N are S.H.M.'s, and taken together they make up the motion of P. These two S. H. M.'s have the same amplitude and period, but when M is

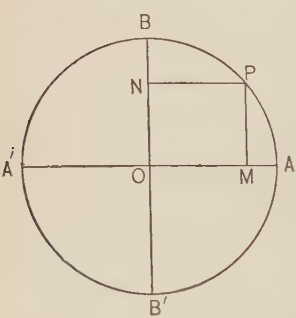

Fig. 20.

at its greatest displacement, that is, at A or A', N is at zero displacement. Hence one motion is one-fourth of a vibration ahead of the other, or the motions differ in phase by one-fourth of a period or by $\frac{1}{2}\pi$. This is also the difference of the epochs, since the epoch of a S. H. M. is its phase at time $t = 0$.

43. Trigonometrical Expression for the Displacement in S. H. M. — If r is the amplitude of a S. H. M. and x the displacement at time t, it is obvious from Fig. 18 that

$$x = r \cos(\omega t + \epsilon).$$

If the vibrating point is at its greatest positive displacement (*i.e.* at A in Fig. 18) when $t = 0$, then $e = 0$ and

$$x = r \cos \omega t.$$

If, on the other hand, M is at zero displacement when $t = 0$ and is moving in the positive direction, then $\epsilon = -\frac{\pi}{2}$ and

$$x = r \cos(\omega t - \tfrac{1}{2}\pi)$$
$$= r \sin \omega t.$$

If we replace ω by $\frac{2\pi}{T}$, T being the period of the S.H.M., we get trigonometrical expression for S. H. M. in which no reference to circular motion appears.

44. The Simple Pendulum. — A simple pendulum consists of a small heavy body (usually a sphere) called the bob suspended by a cord (or wire) whose weight may be neglected compared with the weight of the bob. When set vibrating through a small angle the bob describes a small arc that is approximately a straight line.

Let θ be the angle that the cord makes at any time with the vertical. If allowed to fall vertically the acceleration of the bob would be the acceleration of gravity, g. Since it is confined to the arc the acceleration, a, of the bob in the positive direction of its motion is

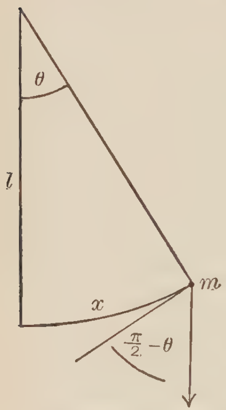

FIG. 21.

$$\begin{aligned} a &= -g\cos\left(\frac{\pi}{2}-\theta\right) \\ &= -g\sin\theta \\ &= -g\left(\frac{\sin\theta}{\theta}\right)\theta \\ &= -\frac{g}{l}\left(\frac{\sin\theta}{\theta}\right)\cdot x, \end{aligned}$$

x being the displacement of the bob from the centre of its path of vibration. If θ be less than 3°, $\left(\frac{\sin\theta}{\theta}\right)$ will differ from unity by less than 3 parts in 10,000. Taking it as unity,

$$a = -\frac{g}{l}\cdot x.$$

Hence the motion is (§ 40) S. H. M. and

$$T = 2\pi\sqrt{-\frac{x}{a}} = 2\pi\sqrt{\frac{l}{g}}.$$

45. Vibrations of a Tuning-fork and of a Weight suspended by a Spring. — It will be shown later that a point on a tuning-fork executes a motion that is very nearly S. H. M. and that the same is true of a mass suspended by a spiral spring and vibrating vertically.

Exercise VI. Study of Motion of Pendulum

A heavy iron cylinder forms the bob of a pendulum, the suspension being a steel rod the mass of which is small compared with that of the cylinder. The ends of the cylinder are parallel to the plane of vibra-

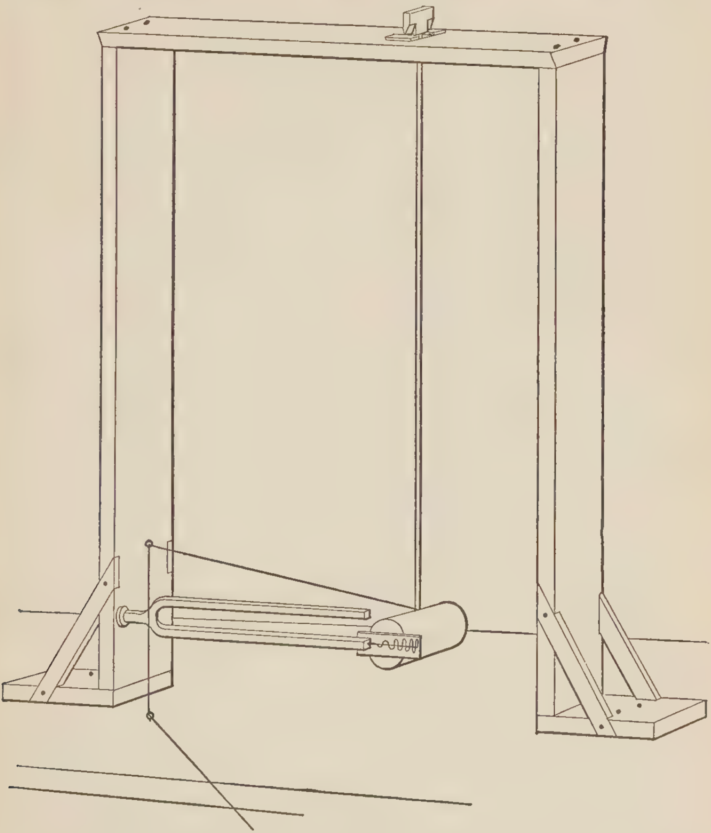

Fig. 22.

tion, and to one end a glass plate, on which a millimetre scale is etched, can be attached by three small clips. A tuning-fork is clamped to the framework that supports the pendulum in such a position that its

prongs vibrate parallel to the plane of vibration of the pendulum. A fine needle-point carried by a light flexible strip of brass that is attached to one prong of the fork presses against the glass plate. If the plate be coated with soap (bon ami) and allowed to dry, the needle-point will trace a clear sharp-cut curve on the glass plate when both pendulum and tuning-fork are in vibration.

The pendulum should be drawn aside about 5 cm. and held by a cord which passes through a couple of screw eyes (as in Fig. 22), and is held by pressure of a thumb on the table; when the thumb is removed the pendulum will be released without any jar. The tuning-fork is started by a blow from a small wooden mallet and then the pendulum is released. To prevent confusion of the record the pendulum should be arrested at the end of half of a complete vibration. When the tuning-fork has stopped the pendulum should be released and allowed to complete the vibration and again arrested. The pendulum should then be allowed to stand vertical and quite at rest and the fork set into a slight vibration so as to give a record of the middle of the arc of vibration.

If the plate be removed from the pendulum and held up in front of a window (or placed at an angle in front of a mirror that rests on the table so as to reflect light from the window) and then examined through a magnifying glass, the length of the successive half-waves of the curve can be read on the etched scale with considerable accuracy. Assuming that each vibration of the tuning-fork is completed in the same time, τ, each group of 3 waves in the record will be the distance the bob of the pendulum travels in time $3\,\tau$. The lengths of the successive groups should be recorded with the greatest possible accuracy. The nature of the motion should then be studied as in Exercise V, the successive groups corresponding to OM_1, M_1M_2 $\cdots$ in that exercise.

From the known frequency of vibration of the tuning-fork and the record just obtained the period of vibration of the pendulum may be calculated by means of the formula in § 39. (See also (g) in the discussion of Exercise V.) The period of vibration should also be calculated by means of the formula for the simple pendulum, the length of the pendulum being taken as the distance between the knife-edges and the centre of the bob. Finally the period should be found

experimentally by counting the number of vibrations in two or three minutes.

(A scale etched on the glass is not indispensable. If the glass is plain, the wave lengths may be read by placing the glass, face down, on a millimetre scale.)

DISCUSSION

(*a*) Sources of error.

(*b*) In what respect is the motion of the pendulum not exactly S. H. M?

(*c*) What form of curve would be obtained if the bob of the pendulum were replaced by a body moving horizontally, (1) with constant velocity, (2) with constant acceleration?

(*d*) What curve would be obtained if the bob of the pendulum were replaced by a body falling freely while the tuning-fork vibrated horizontally?

(*e*) Calculate the length of a second's pendulum.

(*f*) How much shorter than a second's pendulum would a clock pendulum be that lost one minute per day?

(*g*) A pendulum which is a second's pendulum where $g = 980$ vibrates 3599 in an hour at the top of a mountain. Find the acceleration of gravity at that point.

46. The Projection of a S. H. M. on a Straight Line in the Same Plane is also a S. H. M. — Let M be a point having S. H. M. along $A'A$. If the projections of M and A be N and B respectively, then

$$\frac{ON}{OM} = \frac{OB}{OA}.$$

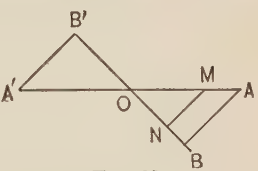

Fig. 23.

But $OM = OA \cos(\omega t + \epsilon)$,

$$\therefore ON = OB \cos(\omega t + \epsilon).$$

Hence the projection of a S. H. M. is a S. H. M. of the same period. By reversing the proof it can be shown that a linear vibration that projects into a S. H. M. is itself a S. H. M.

47. Composition of S. H. M.'s in Lines at Right Angles. — If a point has one S. H. M. in a vertical line and another in a horizontal line, its resultant displacement at any time can be found by compounding its vertical displacement, y, and its horizontal displacement, x, at that time. These might be found from the trigonometrical expressions for

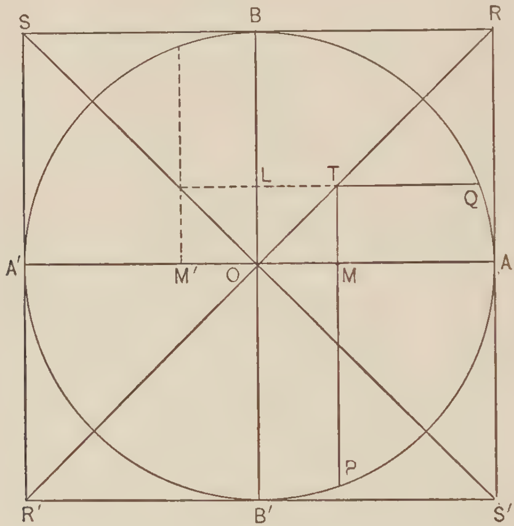

FIG. 24.

the two S. H. M.'s. They can also be readily found by projection from the corresponding circular motion.

(1) The simplest case is when the S. H. M.'s have the same period, amplitude, and phase. Then the circles of reference are coincident. The two S. H. M.'s may be regarded as beginning at O at the same time. The horizontal motion is the projection on $A'A$ of the motion of a point P that revolves in the circle beginning at B', and

the vertical motion is the projection on $B'B$ of the motion of a point Q that revolves in the circle beginning at A. It is obvious that OL will always be equal to OM and that the point T that has both of these motions will always lie in the bisector $R'R$ of the angles AOB and $A'OB'$. Moreover, it is clear that the motion of M is the projection of the motion of T. Hence by § 46 the motion of T is also a S. H. M. of the same period as the components.

(2) When the period and amplitude are equal but the phases differ by π or one-half of a vibration, it can be shown in the same way that the resultant is a S. H. M. along the bisector $S'S$ of the angles $B'OA$ and BOA'. The only difference in the proof consists in supposing P to begin at B instead of at B'.

(3) When the periods and amplitudes are equal and the phases differ by $\frac{1}{2}\pi$ or one-fourth of a period, the motion is uniform circular motion in the circle of reference, as has been already shown in § 42.

(4) When the phase difference is anything else, the period and amplitude still being equal, the path can be shown to be an ellipse inscribed in $RSR'S'$.

(5) Let us next suppose that the periods are equal but the amplitudes different, the amplitude of the vertical motion being the greater. The result will be a relative elongation of all the vertical lines of Fig. 24. Corresponding to (1) and (2) above we shall still have motion in straight lines but the lines will be closer to the vertical, and corresponding to (3) and (4) we shall have motion in ellipses whose directions depend on the phase relations.

(6) If the periods be slightly different, then one motion

will continually gain in phase on the other; at any moment the motion will be in an ellipse whose form and position will depend on the difference of phase at that moment, but the ellipse will be continually changing in form and position, passing through the circle and straight line as particular cases.

48. Composition of S. H. M.'s of Different Periods in Lines at Right Angles. — Let P have a vertical S. H. M., of which the amplitude is 4 cm., represented by OA and the period 3 sec., and a horizontal S. H. M., of which the amplitude is 2 cm., represented by OB and the period 2 sec. To avoid confusion draw the circle of reference, C, of the vertical S. H. M. at a distance from O with its centre on a horizontal line through O and the circle of reference, C', of the horizontal S. H. M. at a distance from O with its centre on a vertical line through O. If C be divided into twelve equal parts and C' into eight equal parts, then the time of describing each part of C and C' will be the same, namely $\frac{1}{4}$ sec. Consider the case in which the vibrating point that has the sum of the two S. H. M.'s is at its greatest displacement, OA, in the vertical S. H. M. when it is at its greatest displacement, OB, in the horizontal S. H. M.

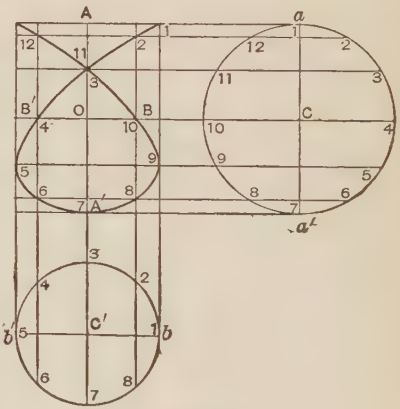

FIG. 25.

Then at time $t = 0$ the resultant position is the intersection of a horizontal line through point 1 of circle C and a vertical line through point 1 of circle C'. At time $t = \frac{1}{4}$ sec. the resultant position is the intersection of a horizontal line through point 2 of C, and a vertical line through point 2 of C', and so on. Thus the curve in which the vibrating point moves can be traced out point by point.

If the phase relation between the two S. H. M.'s be something different from the above, a different resultant curve will be obtained. The different curves obtained by making the vertical motion start $\frac{1}{4}$ sec., $\frac{2}{4}$ sec., $\frac{3}{4}$ sec., etc., later than the horizontal motion are readily drawn.

If the ratio of the periods be slightly different from 3 : 2, the resultant curve will pass continuously through each of these forms and back again.

For other ratios of the period similar sets of curves are obtained, each set being characteristic of a particular ratio of the periods.

Exercise VII. Resultant of S.H.M.'s at Right Angles

A Blackburn pendulum is used in this experiment. An endless braided cord passes over two hooks in the ceiling and is brought together by a small ring attached to one side of the cord, so that the whole suspension has the form of a Y. The bob is a heavy ring of lead soldered to a metallic disk, and carrying a quantity of fine sand, which can stream out through a small hole in the centre of the disk. When allowed to swing, the bob is subject to two S. H. M.'s at right angles, the period of one being invariable and determined by the distance of the bob from the ceiling, while the position of the other depends on the position of the small ring through which the cord passes. The motion of the bob is traced

on a cross-section board by the stream of sand. The sand should be sifted clean and dried by being heated in a dipper before it is used.

The position of the ring should first be adjusted so that the periods are in some simple ratio, say 3 : 4. This will require a few preliminary trials, which, for convenience in collecting the sand, may best be made on a sheet of paper placed below the pendulum. When the proper adjustment has been attained so that the same curve is traced over and over again, the paper may be replaced by the cross-section board placed centrally below the pendulum when at rest. To obtain wide amplitudes and a clear curve, release the bob a couple of inches from the corner of the board, and arrest the motion when the curve has been completed once. Then reproduce the curve on cross-section paper as in Exercise III, leaving the adjustment in the meantime undisturbed.

The same resultant curve should also be derived by graphical composition, as described in § 48. Since the pendulum started from *rest*, the phases of the components were initially the same.

Before the adjustment is disturbed the effect of varying the phase relation should be studied. For this purpose the pendulum may be started with an impulse from various parts of the board. Two or three such variations should be obtained and drawn free-hand on the cross-section paper.

The effect of slightly varying the ratio of the periods may be studied by slightly raising or lowering the small ring. The exact ratio of the periods, if they are commensurate, may be found by counting the number of oscillations in the two directions. Some of the curves corresponding to different ratios of periods, such as 2 : 1, 5 : 3, etc., should be obtained and drawn free-hand, and the effect of varying phase and ratio of period examined as before.

Finally the ring should be drawn as near to the top as possible, and the resultant of two S. H. M.'s of nearly the same period studied.

DISCUSSION

(*a*) Points on the first curve drawn at which the phase difference is 0, $\frac{1}{2}\pi$, π, $\frac{3}{2}\pi$.

(*b*) Does the escape of sand change the period?

(*c*) Ratio of the periods when the ring is halfway between the ceiling and the bob of the pendulum.

(*d*) Where and in what direction might an impulse be given to the pendulum without influencing the *form* of the curve?

49. Composition of S.H.M.'s in the Same Line. — Cases in which a point has two or more S. H. M.'s in the same line occur in the transmission of sound, light, and electrical waves. Each component S. H. M. may be regarded as the projection of a uniform circular motion. Let us suppose that P rotates steadily in a circle and that the motion of the projection, L, of P is one of the component S. H. M.'s. Let another point, Q, rotate uniformly in another concentric circle, and let the motion of the projection, M, of Q be the other component S. H. M. If ON be always equal to the sum of OL and OM, the motion of N will be the sum of the two component S. H. M.'s. ON is evidently the projection of OR, the diagonal of a parallelogram of which OP and OQ are adjacent sides.

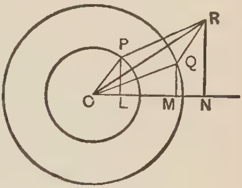

FIG. 26.

(1) Let the periods of the component S. H. M.'s be equal. In this case P and Q move with equal angular velocities and the angle POQ remains constant. Hence R rotates in a circle in the same period as P and Q, and the motion of N is, therefore, a S. H. M. of the same period as the components. The phase of N's motion at any time is the angle RON, and this is always intermediate between the phases of the components or the angles POL and QOM. The amplitude, OR, of the resultant depends on the amplitudes, OP and OQ, of the components and

also on the constant phase difference POQ. When POQ is 0, the components are in the same phase, and OR is the sum of OP and OQ; and when POQ is 180°, OR is the difference of OP and OQ.

(2) Let the periods of the components be different. Let the period of L and P be T_1, and let that of M and Q be T_2, and suppose $T_1 < T_2$, so that the angular velocity of P is greater than that of Q. In this case, as P and Q rotate, the angle POQ increases at a constant rate. Hence R does not move in a circle and the motion of N is therefore not a S. H. M. If T_1 and T_2 are nearly equal, the amplitude of the motion of N varies between the sum of OP and PQ and their difference. The farther T_1 and T_2 are from equality the more rapidly do the variations occur. The rate of increase of POQ is the difference between the angular velocities of P and Q, that is, $\frac{2\pi}{T_1} - \frac{2\pi}{T_2}$. When POQ is zero, the components are in the same phase, and when POQ has increased to 2π, they are again in the same phase; if this requires a time T, the rate of increase of POQ is $\frac{2\pi}{T}$. Hence

$$\frac{1}{T} = \frac{1}{T_1} - \frac{1}{T_2},$$

or, denoting the respective *frequencies* (§ 29) of the components by n_1 and n_2 and the frequency of coincidences by n,

$$n = n_1 - n_2.$$

Exercise VIII. Resultant of S. H. M.'s in the Same Line

Two light spiral springs, each carrying a weight, are placed some distance apart so that the weights are at the same level. A very light wooden rod is suspended from the weights by short threads that leave

the rod considerable freedom of motion. If one weight be kept at rest while the other is in vertical vibration, the centre, C, of the rod will have a S. H. M. of the same period as the vibrating weight but of half the amplitude. When both weights are in vibration the motion of C is the sum of two S. H. M.'s in the same line. A vertical scale placed behind C will show the amplitudes of the vibrations.

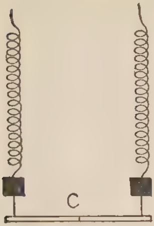

FIG. 27.

If both weights be drawn downwards until the rod is horizontal and then released simultaneously, they will start in the same phase of vibration (that of greatest displacement). When the rod is horizontal and for a moment moving parallel to itself, the motions will be again in the same phase and the time between these two coincidences will be the "coincidence period," T. To find T by observation, release the weights on a tick of the clock at the beginning of a minute and find the number of minutes and seconds required for several coincidences. This should be done (1) with such weights and springs that T_1 and T_2 are quite different (*e.g.* about 3 : 2). (2) When T_1 and T_2 are not far from equal. In the first case the coincidences will occur frequently and a large number should be counted and timed.

In each case T_1 and T_2 should be found by observing several times the number of vibrations in three or four minutes and taking the mean. From these T should be calculated (§ 49). T_1 and T_2 will need to be observed with special care in the second case above, since their difference is used in calculation.

Using a scale pan and weights for one suspended body, adjust the weights in the pan until T_1 and T_2 are equal, and then verify the statements of § 49 as regards the composition of S. H. M.'s of the same period; vary both the phase difference and the amplitudes of the components, and note the amplitude of the resultants.

DISCUSSION

(*a*) What is the resultant of three or more S. H. M.'s of the same period in the same line?

(*b*) If two S. H. M.'s of the same period in the same line differ in phase by $\frac{1}{4}$ of a period (or 90°), what is the amplitude of the resultant?

(*c*) Find an expression for the amplitude of the resultant of two S. H. M.'s of the same period and with a phase difference of θ.

(*d*) When the components in (*c*) are of different amplitudes, to which is the resultant nearer in phase?

(*e*) If the coincidence period of two S. H. M.'s is observed and the period of one of them is known, how can that of the other be found ("coincidence method" of rating a pendulum)?

REFERENCES FOR CHAPTER IV

Watson's "Text-book of Physics," Book I, Chapter VII.

Daniell's "Principles of Physics," Chapter V.

Clifford's "Kinematic," Chapter I.

Macgregor's "Kinematics and Dynamics," Part I, Chapter IV.

DYNAMICS

CHAPTER V

FORCE

50. So far we have considered the motion of bodies which have certain velocities and accelerations without inquiring how these velocities and accelerations are produced. Clear, systematic views as to the way in which one body may influence the motion of another body were first arrived at by Sir Isaac Newton. His three Laws of Motion lead to results that have been verified in innumerable cases and this is the ground of our belief that they are accurate. When once stated and understood they seem so nearly obvious that they are sometimes called axioms. Improvements in the way of arranging and stating them will no doubt come in course of time, but they form at the present time the most convenient and satisfactory basis of Dynamics.

51. Newton's First Law of Motion. — "Every body continues in its state of rest or of uniform motion in a straight line except in so far as it is compelled by external force to change that state."

Our primary conception of force is a certain muscular sensation associated with any attempt to move a body or change its motion. When we see the same effect produced in some other way, for example by the impact of a second

body on the first, we attribute the result to a force exerted by the second body on the first. This is a somewhat artificial but very convenient extension of our primary conception of force. A definition of force in this sense is implied in Newton's First Law.

Force is any action between two bodies that changes the motion of either. This must be understood as a definition of the *meaning* of force; a definition of the *measure* of force follows from Newton's Second Law. When a body at rest begins to move or when its motion varies in either magnitude or direction, the effect can be traced to some other body which is said to exert a force on it. The property in virtue of which a body not acted on by any force remains at rest or in uniform motion is called *inertia.*

52. Newton's Second Law consists essentially of two statements: (1) the ratio of two different forces is the ratio of the accelerations they produce in the same body; (2) the ratio of the masses of two different bodies is the inverse ratio of the accelerations produced in them by a certain force. The first enables us to compare and measure forces; the second enables us to compare and measure masses; both are usually combined in a single statement. In giving it we shall use two terms that have become current since Newton's time. The product of a force by the time it acts is called the *impulse* of the force. The product of the mass of a body by its velocity is called the *momentum* of the body.

"Change of momentum is proportional to the impulse of the force applied and takes place in the direction of the force." If a force, F, act for a short time, t, on a body of

mass, m, and if the velocity of the body change from v to v',

$$Ft \propto (mv' - mv),$$

or

$$F \propto m\frac{v' - v}{t},$$

or

$$F \propto ma,$$

a being the acceleration of the body.

The law refers only to the *change* of momentum produced by a force and therefore implies that the change of momentum is the same no matter what the initial motion of the body. It also implies that the change of momentum produced by a force is independent of the action of other forces, or when a number of forces act on a body we may calculate their results independently and then compound these results.

53. Units of Mass and Force. — Newton's Second Law may be stated thus:

$$F = k \cdot ma,$$

k being a constant the value of which depends on the units of mass and force adopted (it being understood that the unit of acceleration is fixed by the units of length and time already chosen). If suitable units of mass and force be chosen, k will be unity. Let the unit of mass be chosen first and then let the *unit of force* be taken as *that force which acting on unit mass gives it unit acceleration.* Since in this case F, m, and a are all unity at the same time, k must also be unity and therefore

$$F = ma.$$

The unit of mass that is usually employed in Physics is the mass of a thousandth part of a certain block of platinum-iridium kept at Sèvres near Paris and known as the *kilogramme prototype.* The thousandth part of this mass is

called the *gramme*. The corresponding unit of force, or the force that would give a gramme an acceleration of 1 cm. per second per second is called the *dyne*. The gramme and the dyne are the units of mass and force respectively in the absolute C. G. S. (centimetre-gramme-second) system of units. The gramme is (*very* nearly) the mass of 1 cc. of water at 4° centigrade.

54. Mass and Weight. — Every body on the surface of the earth is attracted by the earth with a certain force called the weight of the body. Newton showed that at any one place the attractions on different bodies are proportional to the masses of the bodies. The experiments by which he proved this consisted in timing pendulums of the same length but with bobs of different sizes and different materials. He found that they all vibrated in the same time. Two such pendulums when at the same inclination to the vertical are acted on in the direction of motion by the same fraction of the force of gravity. But since they vibrate in equal times they must at equal inclinations to the vertical have the same acceleration in the direction of motion. Therefore the ratio that the force in the direction of motion bears to the mass must, by Newton's Second Law, be the same for the two pendulums. Hence the whole forces of gravity on the bobs must be proportional to their masses, or *weight is proportional to mass*. This is the basis of the most convenient method of comparing masses, namely, by comparing the weights of the masses by means of a balance.

While there is this close connection between mass and weight, it must not be forgotten that mass or inertia is essentially different from weight or the force of gravity.

At a very great distance from other attracting bodies the weight of a body would be very small, while its mass is everywhere the same.

If m be the mass of a body and W its weight in absolute units of force,

$$W = mg,$$

g being the acceleration of gravity, which is the same for all bodies at the same part of the earth.

55. Gravitational Unit of Force. — In dealing with problems in which weight is the chief force to be considered, engineers find it convenient to use the weight of a pound (in English-speaking countries) as their unit of force, the mass of a pound being the unit of mass. In this case the value of k in Newton's Second Law cannot be unity. For, if the unit of force (a pound weight) act freely on the unit of mass (a pound), the acceleration produced is g.

$$\therefore\ 1 = k \cdot 1 \cdot g, \text{ or } k = \frac{1}{g}.$$

Hence, in this case, the formula for Newton's Second Law is

$$F = \frac{1}{g} ma,$$

it being understood that F is measured in the weight of a pound as unit of force, and m in the pound as unit of mass. The inconvenient factor $\frac{1}{g}$ may be omitted if a mass of g-pounds be taken as unit of mass.

Exercise IX. Force and Acceleration

Apparatus. — A bicycle wheel is mounted on top of a tall post and a cord carrying large iron masses is stretched over the wheel. In the wood of the rim (and somewhat closer to one side to avoid the spokes)

a V-shaped groove is turned, and the cord rests in this groove. A simple form of clamp fixed to the post enables the operator to keep the wheel fixed; a slight jerk on a cord attached to the clamp will release the wheel. If the axis of the wheel passes through the centre of gravity of the wheel and also through the centre of the groove in which the cord rests, the wheel will have no tendency to move when the masses at the end of the cord are equal. A small weight placed on one of the large masses will cause the latter to move with a constant acceleration when the wheel is released. Additional large masses and different small weights are provided so that the masses moved may be changed and also the forces causing the motion varied. The observer can note the position of the large masses at any time by reading a vertical scale which stands close behind one of them. The cord sustaining the masses extends to the floor on both sides, so that its weight on each side is always the same.

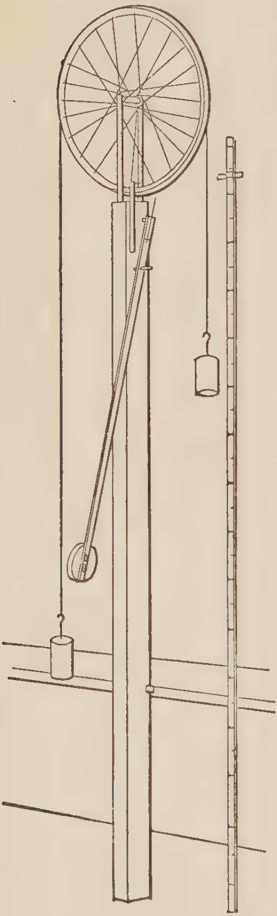

FIG. 28.

Adjustments. — The supporting post must be adjusted until the cord has no tendency to move out of its groove. On four of the spokes of the wheel there are small movable weights which must be adjusted until the wheel remains at rest in any position when the masses supported by the cord are equal.

Distance and Acceleration. — When the acceleration in the line of motion is constant, the distance traversed from rest is proportional to the square of the time. This may be tested by attaching equal masses to

the cord (preferably the largest masses supplied) and placing a small weight on one. One mass being elevated to a position observed on the vertical scale and the other being near the floor, the wheel is released on a tick of the clock and the position of the descending mass at the third succeeding tick noted (the intermediate seconds being passed over, since the distances are too small to be accurately observed). This should be repeated twice and the mean of the three readings taken. The same should be done for each succeeding second, until the descending mass reaches the floor. Twice the distance in each case divided by the square of the time should give a constant, namely, the acceleration.

Device for Recording Distances. — In the above, distances of descent are supposed to be observed by eye. This has the advantage of simplicity, and with care gives satisfactory results. If preferred, a record of the motion may be obtained by the following device. A heavy, adjustable pendulum, adjusted to beat seconds, swings from a knife-edge attached to the post. The rod of the pendulum extends above the knife-edge, and to the upper end of the extension a small camel's-hair brush is attached. The brush is inked once in a vibration by touching a wet stick of India ink, and as the pendulum passes through the vertical, the brush makes a trace on a strip of mucilaged paper wrapped around one side of the rim of the wheel. When the pendulum is released at the beginning of an experiment, it releases the wheel on first passing through the vertical and at the same time makes the first record on the paper.

Since the movement of the paper keeps pace with the movements of the weights, the record on the paper is a record of the movements of the weights. The paper is removed at the end of an experiment and its record interpreted.

Newton's Second Law of Motion. — (1) The accelerations given to a mass are proportional to the forces applied. This may be tested by comparing the acceleration just measured with the acceleration when the weight placed on the descending mass is doubled. All of the readings need not be repeated; it will be sufficient if the distance for one particular number of seconds is determined, preferably the largest number for which observations can be conveniently made.

(2) The accelerations produced by a given force are inversely as the masses set in motion. This may be tested by replacing the masses attached to the cord by others half as great and finding the acceleration as before.

Calculation of Distances. — From the masses, the value of g and the greatest time observed in each of the preceding cases, the corresponding distances should be calculated. Any discrepancy between the observed and the calculated distances should be accounted for.

Tension of Cord. — The equation for the motion of each mass should be stated and the tension of the cord deduced in each of the three cases studied.

(This exercise will be continued from the point of view of energy and angular motion of the wheel in Exercise XXV.)

DISCUSSION AND PROBLEMS

(*a*) Meaning and deduction of formulæ used.

(*b*) Is the tension the same in all parts of the cord?

(*c*) Effect of friction.

(*d*) Calculate what addition to the large masses would be equivalent to the mass of the wheel.

(*e*) Calculate for one of the cases studied the height to which the ascending mass rises after the other mass strikes the floor.

(*f*) Suppose that in one of the cases studied the cord were to break when the masses are at the same level. (1) What would be the interval between the impacts of the masses on the floor? (2) With what velocities would they strike the floor?

(*g*) Twelve bullets are divided between two scale pans connected by a cord passing over a very light pulley. What division of the bullets will produce the greatest tension of the support of the pulley?

(*h*) A cord passes over two fixed pulleys and through a third pulley suspended between them. A mass of 10 kg. is attached to one end of the cord, a mass of 5 kg. to the other end, and the suspended pulley and an attached weight weigh 2 kg. The parts of the cord being vertical, with what acceleration will the masses move if released?

56. Force of Gravitation. — The weight of a body is a particular case of the attraction between bodies called *gravitation.* From a study of the motions of the moon and the planets, Newton discovered that the accelerations of these bodies are due to the fact that between any particle of mass m_1 and another particle of mass m_2, at a distance r from the first, there is an attraction expressed by the formula

$$F = G\frac{m_1 m_2}{r^2},$$

G being a constant called the *constant of gravitation.* So far as known this law is perfectly exact. It is true for bodies like the planets and the sun at great distances apart, and very careful experiments have shown that it is true for small bodies only a few centimetres apart. Whether it also holds true for much smaller distances is not yet known. Experiments have shown that the attraction between two bodies does not depend on the materials of which they consist and is not influenced by intervening bodies.

Newton also showed that a body of spherical shape, and of the same density at all points equally distant from the centre, attracts external bodies as if it were concentrated at the centre. The earth is very nearly such a body, and the attraction between it and a body outside of its surface varies nearly inversely as the square of the distance of the body from the centre of the earth. But the earth is not quite spherical, and a body on the surface of the earth is farther from the centre the nearer it is to the equator. This slightly affects the acceleration, g, of a falling body, and there is also an effect due to the rotation of the earth

(§ 64) and different in different latitudes. Measurements of g by the pendulum agree fairly closely with the formula

$$g = g_0(1 - .0026 \cos 2\lambda - .0000003\, l),$$

where λ is the latitude, l the height above sea-level in metres, and $g_0 = 980.6$.

The value of the constant of gravitation, G, has been found by measuring the attraction between two bodies on the surface of the earth. When m_1, m_2, r, and F are expressed in C. G. S. units, G is 6.6576×10^{-8}. If we use this value for G and consider the attraction between a *gm* and the earth, that is, give m_1 the value 1, r the value of the radius of the earth in centimetres, and F the weight of a *gm* in dynes at a pole or 978, the value of m_2 deduced from the formula for the law of gravitation will be the mass of the earth. This divided by the known volume of the earth gives the mean density of the earth, which is thus found to be 5.527. (Poynting and Thomson's "Properties of Matter," Chapters II and III.)

57. Force in Simple Harmonic Motion. — Since the acceleration of a body having a S. H. M. is

$$a = -\left(\frac{2\pi}{T}\right)^2 \cdot x,$$

the force acting on the body when the displacement is x is

$$F = -m\left(\frac{2\pi}{T}\right)^2 \cdot x.$$

Hence if a body performs a vibrating motion under the action of a force which is proportional to and in the opposite direction to the displacement, the motion is S. H. M., and if the force at a certain displacement is known, the period T can be calculated. When an elastic body such as a spiral spring or a bar is distorted in any way, that is, stretched, bent, or twisted, etc., the force with which it resists the distortion and tends to recover its form is pro-

portional to the distortion (provided the distortion is not so great as to cause a permanent change). This is an experimental fact known as Hooke's law of elasticity (§127). Hence such a body when distorted and set free to vibrate performs S.H. vibrations, the period depending on the force resisting distortion and the mass set into vibration. The vibrations of a spiral spring carrying a weight and of a tuning-fork are examples that have been employed already.

Exercise X. Force in S.H.M.

A mass m is suspended by a vertical spiral spring and vibrates in a vertical line. The motion is S.H.M. if the resultant force acting on the body at any displacement x from its position of rest is proportional to x. Let the length of the spring when the mass is at rest be l. The force F_1 required to stretch the spring to the length l is the weight of m. When m is displaced through a distance x (positive downward) the resultant (upward) force acting on m is $F_2 - F_1$, if F_2 be the force required to stretch the spring to the length $l + x$. Hence the motion will be S.H.M. if $(F_2 - F_1) \propto x$, *i.e.* if $\frac{(F_2 - F_1)}{x}$ = a constant. If various values are given to F_2 and the corresponding values of x noted, a curve connecting F_2 (as ordinate) and x (as abscissa) may be drawn and, if $(F_2 - F_1) \propto x$, the curve will be a straight line. From this line a more accurate value of the constant ratio of $(F_2 - F_1)$ to x can be found and used to calculate the period of vibration of m.

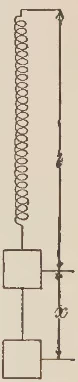

FIG. 29.

The curve thus obtained expresses the relation between the length of the spring and the force applied to it, and is called the "calibration curve" of the spring. When a spring has been calibrated, it may, along with its calibration curve, be used as a spring balance to weigh bodies or to apply known forces to bodies, and as such we shall have frequent occasion to use it.

The calibration of each spring consists in determining the length

for each of half a dozen or more different weights attached and then plotting a curve with stretching forces as ordinates and lengths as abscissæ. The calibration of a spring will be found necessary in several other exercises. A device that facilitates the calibration is a vertical scale etched on mirror glass or on nickel-plated steel. If the spring be hung in front of the glass scale, its length between the hooked ends can be read by reflection without danger of parallax. A numbered tag should be attached to each spring calibrated, and the number should be marked on the calibration curve.

The periods of vibration of two masses attached to spiral springs are to be calculated by the above method and then determined experimentally by counting the number of vibrations in several minutes.

DISCUSSION

(*a*) Limit to the amplitude if the motion is to remain S. H. M.

(*b*) Should the masses of the springs be taken account of in the calculation?

(*c*) How could the vibrations of a spring, carrying a weight, be used to find the value of g?

58. Composition and Resolution of Forces acting on a Particle. — Every force has a definite direction and a definite magnitude. Hence any number of forces acting on a particle can be represented by lines in the directions of the forces and proportional in lengths to the magnitudes of the forces.

Since the forces give rise to accelerations which are in the directions of the forces and proportional in magnitude to the forces, a set of lines that represent any number of forces applied to a particle may be also taken to represent the accelerations to which the forces give rise. These accelerations can be compounded and resolved by methods already stated (§ 9). Hence forces can be similarly compounded and resolved by means of the lines that represent them.

Exercise XI. The Composition of Forces

Composition of Two Forces. — Two spiral springs are attached to a small ring. The other ends of the springs are tied to cords which

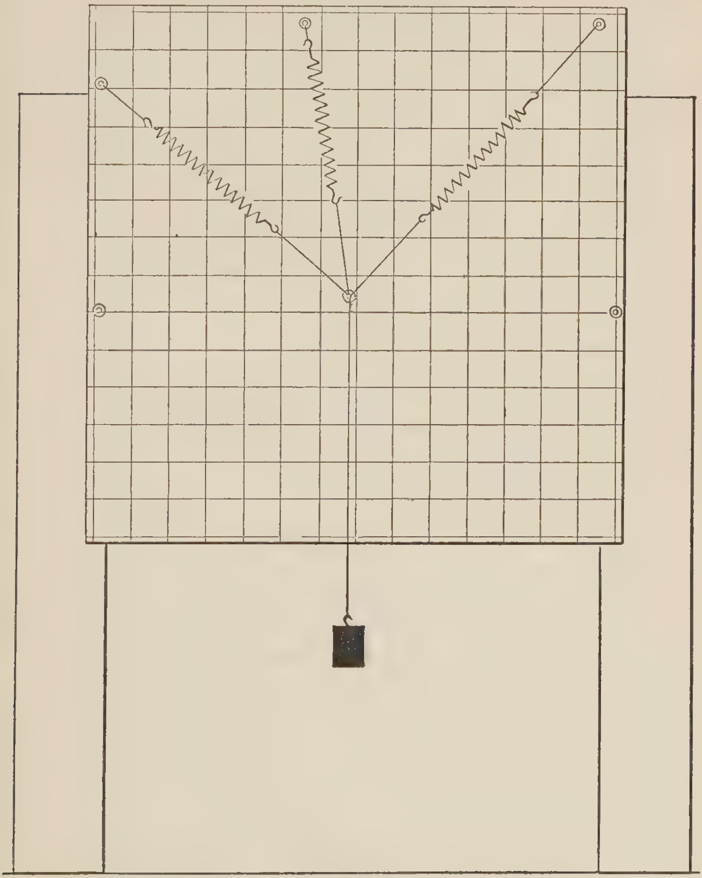

FIG. 30.

are fastened by thumb-tacks to a vertical cross-section board. A weight is hung from the ring by means of a cord. The board may

be levelled by using the weight and cord as a plumb-line. When the ring has come to rest the projection of its centre on the board is marked by a pin. The length of each spring is obtained from a mirror scale placed between it and the board.

A careful copy of the arrangement should then be made on cross-section paper. The direction of the line representing each spring is fixed by the position of the peg to which it is attached and the centre of the ring. If the springs have been already calibrated, the forces they apply to the ring can be deduced from their measure lengths. If they have not been calibrated, each must be hung vertically and the weight required to stretch it as in the experiment determined. The resultant of the forces applied to the springs can then be found graphically by completing the parallelogram. It should also be calculated by the trigonometrical formula. The resultant should be approximately equal and opposite to the weight carried by the cord.

Composition of Three Forces. — An additional spring is attached to the ring and fastened to the board by a peg. A drawing is made on cross-section paper as before. The resultant is then found (1) by the polygon method, (2) by the analytical method, the angles the forces make with the horizontal being measured by a protractor. The resultant should be approximately equal and opposite to the weight carried by the cord attached to the ring.

DISCUSSION

(*a*) Meaning of resultant and proof of formulæ used.

(*b*) What is the sum of the vertical forces on the pegs equal to? Of the horizontal forces?

(*c*) Calculation of the angle between the cord and each spring in the first part of the exercise.

(*d*) Each of the three forces in the first part of the exercise is proportional to the sine of the angle between the other two.

(*e*) How does the tension in each spring vary with the inclination of the spring to the vertical?

(*f*) What is the minimum strength of a wire that will sustain a heavy picture if the angle between the two parts of the wire be 90°?

(*g*) On what does the pull on a kite-string depend?

59. Condition of Equilibrium of Forces acting on a Particle. — Any number of forces acting on a particle are said to be in equilibrium when their resultant is zero.

Two forces are in equilibrium when they are equal and opposite, and they cannot be in equilibrium unless they are equal and opposite.

Three forces are in equilibrium if the resultant of two of them is equal and opposite to the third. If the three forces can be represented by the three sides AB, BC, CA of a triangle, the sides being taken in continuous order, then the resultant of two of the forces AB and BC is equal and opposite to the third CA, and the forces are therefore in equilibrium.

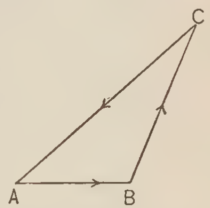

FIG. 31.

Conversely, if three forces are in equilibrium, and if any triangle be drawn whose sides taken in order are in the directions of the forces, then the forces are proportional to the sides of this triangle. For if any two lines AB, BC be drawn to represent two of the forces, the resultant of these two is represented by AC. Hence for equilibrium the third must be represented by the third side CA of the triangle ABC. Any other triangle whose sides are parallel respectively to the sides of the triangle ABC, that is, in the directions respectively of the forces, is similar to ABC. Hence the forces are proportional also to the sides of this second triangle.

It can be shown in the same way that a necessary and sufficient condition for the equilibrium of any number of forces is that they should be representable by the sides of a closed polygon taken in order.

Another convenient way of stating the condition for the equilibrium of any number of forces is supplied by the analytical method of composition (§ 15). If the sums of the components of the forces in three directions at right angles are X, Y, Z, and if R is the resultant,

$$R^2 = X^2 + Y^2 + Z^2.$$

Hence R is 0 if X, Y, and Z are each 0.

Conversely, if R is 0, X, Y, and Z must each be 0, since their squares cannot be negative.

Exercise XII. The Triangle of Forces

An interesting illustration of the triangle of forces is its application to the calculation of the forces that act on the parts of a jointed framework. As a simple example, we may consider a skeleton framework ABC made up of three spiral springs suspended from the vertical cross-section board by two other springs which are attached to the corners B and C of the triangle. A weight W attached to the corner of A will put all the springs in a state of tension. Let the tension in the sides of the triangle and the forces applied to it be T_1, T_2, T_3, F_1, F_2, F_3, as indicated in the diagram.

At the point A three forces T_1, F_1, and T_2 act so as to keep the point in equilibrium, and they may therefore be represented by any triangle oab whose sides are, taken in order, in the direction of the forces. The forces T_2, F_2, T_3, that keep the point B in equilibrium, may similarly be represented by the sides obc of a second triangle which has one side ob in common with the first triangle. Finally, by joining c and a we have a triangle oca whose sides represent the forces T_3, F_3, T_1, which act at C. The figure $oabc$ is sometimes called the "force-diagram" of the framework ABC. From it the magnitudes of all the other forces can be deduced by proportion if that of one of the forces be known.

Having carefully constructed the force-diagram on cross-section paper, assume F_3 as known from the magnitude of the suspended weight, and then deduce the magnitudes of the other forces and tabu-

late the results and the numbers of the springs. As a check on the results, carefully measure the lengths of the springs and tabulate the results, and then deduce the tension of the calibrated springs from

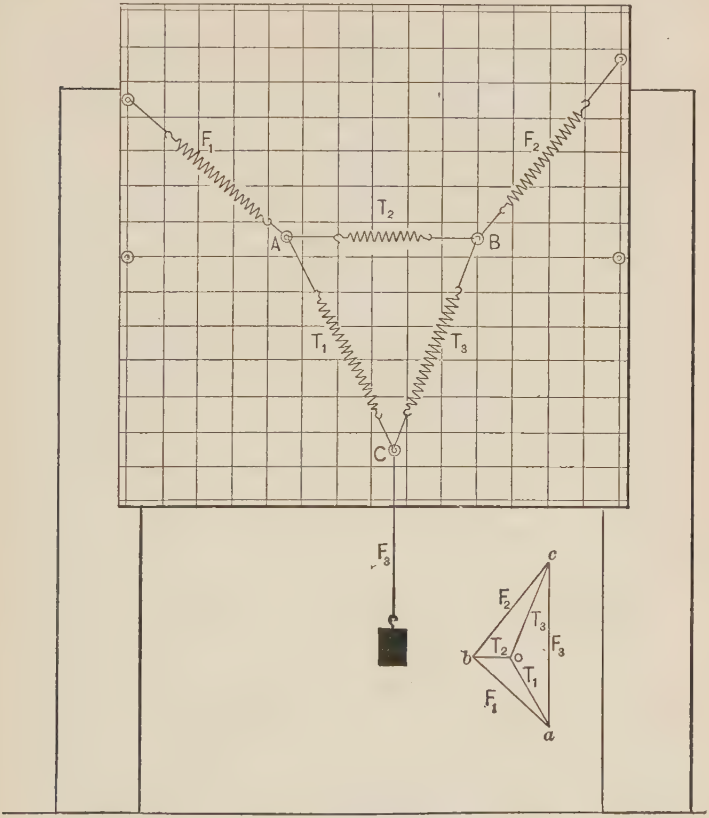

FIG. 32.

their calibration curves. Calibrate the remaining springs, or at least find the forces necessary to stretch them as in the experiment. The closeness of agreement of the results found graphically, and those deduced from calibration of the springs, will depend chiefly on the care with which the force-diagram was drawn.

DISCUSSION

(*a*) Show that the external forces applied to the framework would be in equilibrium if applied to a particle.

(*b*) Show the same for the internal forces in the framework.

(*c*) If the sides of the framework consisted of uniform rods of considerable weight, in what way would the force-diagram have to be modified?

(*d*) A cord fastened at the ends to a support carries weights at various points. Draw the force-diagram of the arrangement (called a funicular polygon).

(*e*) Construction and calibration of a simple form of light balance for letters, etc., on the principle suggested by *d*.

(*f*) Three forces acting on a particle are represented by the sides $\overline{AB}$, $\overline{AC}$, $\overline{BC}$ of a triangle. Find the resultant.

60. Newton's Third Law of Motion. — "To every action there is an equal and opposite reaction or the mutual actions of bodies are equal and opposite."

The truth of this law is readily recognized in cases in which the bodies are at rest; for instance, when one hand is pressed against the other, when a hand is pressed against a wall, when a hand supports a weight, when two equal masses hang by a cord that passes over a pulley, when a horse exerts force on a rope attached to a canal boat which is prevented from moving.

When applied to bodies in motion the meaning of the law is not so obvious. Consider the case of the horse and the canal boat when both are in motion with constant velocity. If the boat did not pull backward on one end of the rope with a force of the same magnitude as that with which the horse pulls forward on the other end, there would be a resultant force on the rope and it would move with an acceleration.

Next suppose the horse and boat are moving with an acceleration. If the rope is so light that its mass may be neglected, then, if there were an appreciable difference in the magnitudes of the forces at its ends, it would move with a very great acceleration. If the mass of the rope is not negligible, we cannot any longer regard the horse and the boat as bodies acting and reacting directly on one another, for now there is a body of definite mass between them. Consider, however, a part of the rope so short that its mass may be regarded as negligible; the pulls at its ends must be equal, for otherwise it would move with a very great acceleration.

From what has been stated it will be seen that (1) the action and reaction spoken of are not two forces acting on the same body, but the action is a force applied to one body, the reaction, a force applied to the other body; (2) the bodies referred to are bodies directly in contact, although, when there is no relative acceleration, or when there is an acceleration but the mass of the intervening connection is negligible, bodies not directly in contact may be treated as if they were in contact.

The reader should consider all the actions and reactions in the case of the horse and the canal boat (1) between horse and ground, (2) between horse and rope, (3) between rope and boat, (4) between water and boat, (5) between water and ground.

61. Stress. — Forces always occur in pairs, an action and a reaction. The action and reaction considered together are called a *stress*. A force is only a partial aspect of a stress, that is, a stress considered only as regards its

action on a single body. The complementary aspect of the stress is the reaction on the other body. In terms of stress Newton's Third Law may be stated thus: "All force is of the nature of stress; stress exists only between two portions of matter, and its effects on these portions are equal and opposite."

There is reason to believe that when two bodies seem to influence one another's motion without any visible connection existing between them, *e.g.* two magnets or two bodies charged with electricity, the effect is really due to a stress in an intervening medium. In the case of magnetized and electrified bodies, the medium is the *ether* and the nature of the stresses are to some extent understood. In one important case, namely, the gravitational attraction between bodies, the nature of the stress has not yet been discovered, but the ether is probably the medium.

62. Transference and Conservation of Momentum.—The force that a body A exerts on a body B is equal and opposite to the force that B exerts on A. Hence the change of momentum that A produces in B in any time is equal and opposite to the change of momentum that B produces in A in the same time. If, therefore, we reckon momentum in one direction as positive, and in the opposite direction as negative, the mutual action between two bodies produces no resultant change of momentum; one suffers a decrease of positive momentum, the other an increase. Hence momentum may be transferred from one body to another, but the total momentum is unchanged by the mutual action. This principle is sometimes called the *conservation of momentum.*

63. Changes of Velocity due to Mutual Action between Bodies. — When two bodies act on one another the changes of momentum produced are equal in magnitude; hence the changes of velocity are inversely as the masses. If the bodies undergo equal changes of velocity, their masses are equal. If the changes of velocity are unequal, the masses are inversely as the changes of velocity. These statements might be taken as definitions of equality of masses and the ratio of two masses. They are in reality the same as the definitions supplied by Newton's Second Law (§ 52). This way of defining the ratio of two masses leads at once to an experimental method of ascertaining the ratio of the masses; and while it is not an accurate practical method, an attempt to carry it out will help to make the meaning of mass more definite.

Exercise XIII. Transference and Conservation of Momentum

Two light wooden trays or carriers are suspended by threads so as to be free to vibrate while remaining always horizontal. If drawn aside and released simultaneously, they collide perpendicularly at their lowest positions and two needle-points attached to one of them stick into the other and so prevent separation. Small scales are mounted on a bar below the carriers and pointers attached to the carriers move along the scales as the carriers swing.

The threads should be carefully adjusted so that each carrier moves parallel to the line of the scales and so that the carriers just come into contact when hanging at rest. A convenient method of adjustment is to provide the ends of each thread with small rings. One ring is attached to a hook on the carrier, the other is fastened to the upper surface of the top-board by a thumb-tack. After all the threads have been carefully adjusted side-strips on the top-board are screwed down so as to prevent the threads getting out of adjustment.

The carriers can be released simultaneously by means of a thread

that passes through four screw-eyes (see Fig. 33) and is attached to both carriers. The carriers having been drawn aside to any desired

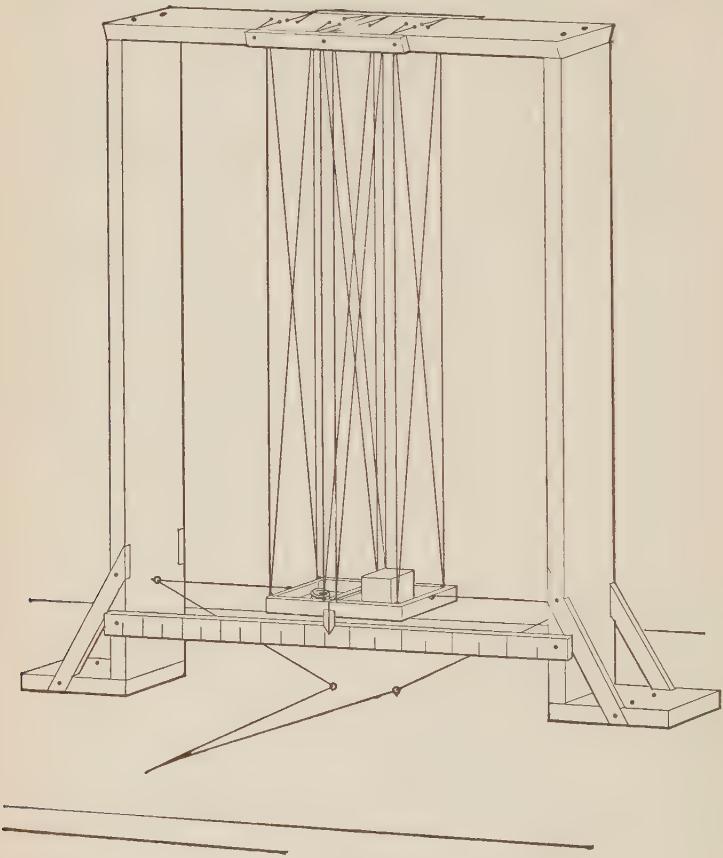

FIG. 33.

positions by means of the thread, the latter is attached to the table by a pin; when the pin is pulled out the carriers are released.

The reader will have no difficulty in showing that if R be the radius of the circles in which the carriers swing and x the horizontal

distance of the starting point of a carrier from the lowest part of the arc, its velocity at the lowest point is $v = x\sqrt{\frac{g}{R}}$. Hence $\sqrt{\frac{g}{R}}$ may be calculated once for all; then v can be found for any observed value of x.

Measurement of Mass. First Method. — The body of unknown mass (a cylinder of wood) is placed in one carrier and known masses ("weights" from a box of weights) are placed in the other carrier until the carriers, falling from the same height, come to rest on colliding. This should be repeated several times, the height of fall being varied and the known and unknown masses interchanged. The accuracy of the result should be tested by placing the unknown and known masses on the pans of an ordinary balance.

Measurement of the Mass of a Carrier. — Let a known mass be placed in one of the carriers and let the carriers be released from such heights that they come to rest on colliding. Then let the known mass be placed on the other carrier and the experiment repeated to ascertain whether the masses of the carriers are equal. Make several careful determinations of the mass of each carrier.

Measurement of Mass. Second Method. — Suppose only a single known mass is available. Place the body of unknown mass in one carrier and the known mass in the other and find the heights from which the carriers must be released so that they shall come to rest on impact. Interchange and repeat.

Measurement of Mass. Third Method. — Place the unknown mass in one carrier and a known mass in the other. Allow the carriers to fall from equal heights and find the velocity of the combined mass after collision. Next let one carrier impinge on the other at rest. Other combinations of initial velocities may be tried. From each the unknown mass is calculated.

DISCUSSION

(*a*) Sources of error.

(*b*) Meaning of equality of mass.

(*c*) Meaning of ratio of two masses.

(*d*) Proof of formula used in calculating v.

(*e*) Is the present method of comparing masses independent of the assumption that weight is proportional to mass?

(*f*) On what ground is it assumed that the carriers, if released at the same time, always meet at the lowest points of their swings?

(*g*) Measurement of the velocity of a bullet by attaching the gun to a heavy pendulum and noting the deflection of the pendulum when the gun is fired.

64. The Force required to make a Body revolve in a Circle. — We have already seen (§ 33) that when a body revolves in a circle of radius r with linear speed s it has an acceleration $\frac{s^2}{r}$ toward the centre. To give it this acceleration, a force directed toward the centre must be applied to it. Since $F = ma$, and $a = \frac{s^2}{r}$, the necessary force toward the centre is

$$F = \frac{ms^2}{r}.$$

To this force there is an equal and opposite reaction due to the inertia of the body. This reaction of a revolving body against acceleration toward the centre is called "centrifugal force." *It must not be thought of as a force acting on the body;* the only force acting on the body is toward the centre, the "centrifugal force" is the reaction of the body in a direction away from the centre. Thus when a stone is whirled around at the end of a string, the force applied to the body is a pull toward the centre produced by the hand; the "centrifugal force" is the outward pull the body exerts on the hand.

Exercise XIV. Acceleration and Force in Uniform Circular Motion

Apparatus. — A horizontal spiral spring connects a body that rotates in a horizontal circle to a vertical steel axis supported on needle-points. The weight of the rotating body, a lead block, is borne by a cord attached to a horizontal rod that passes through the steel axis.

The cord is given the form of a V in order to keep the body from swaying backward and forward in the arc of the circle described by the body. When the vertical axis rotates steadily, the spring is stretched and the lead block moves in a circle. If the plane of the

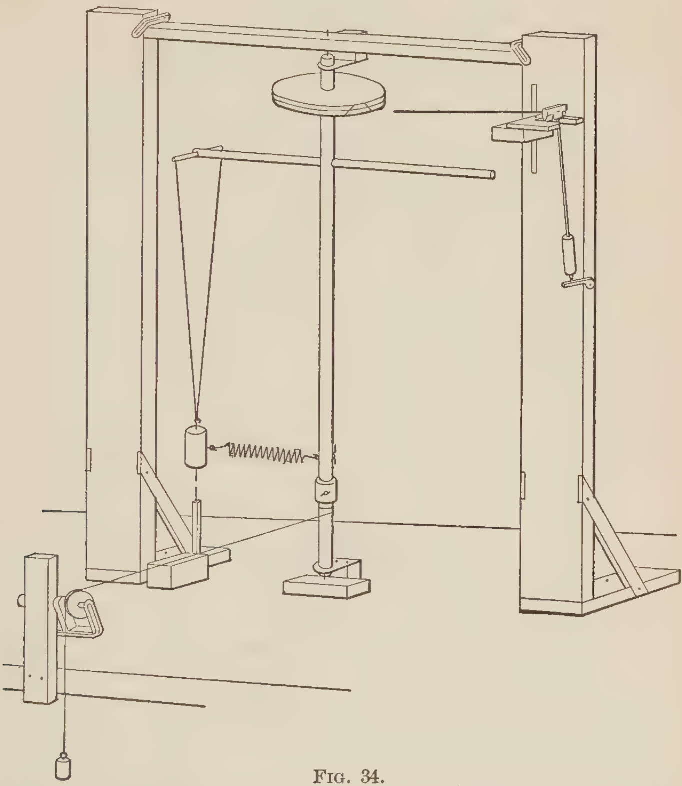

FIG. 34.

cord be vertical, the horizontal central force acting on the lead block will equal the tension of the spring.

The rotation of the axis is produced by a silk thread that is wrapped around the axis; the thread passes over a pulley and carries a weight. The thread is attached to a collar that is adjustable along

the axis; a small ring attached to the thread hangs on a peg attached to the collar, so that when the thread is wholly unwrapped it becomes detached and then the axis rotates at a constant rate (except for the small effect of friction).

For recording the speed of rotation at any time the axis carries a horizontal circular disk around which a strip of paper is fastened. A fine-pointed camel's-hair brush attached to the knife-edge of a short adjustable pendulum sweeps across the strip of paper as the pendulum vibrates; when inked the brush records the vibrations of the pendulum on the strip of paper.

Adjustments. — After the weight has been hung in position and before the spring is attached, a pin is fixed vertically in the bottom of the lead block. Another pin is fixed in the top of a wooden block that rests on the table just below the lead block. The wooden block is placed on the table so that the pins are in the same vertical line. The spring is then attached. It should be horizontal when so stretched that the pins are opposite one another. This adjustment can be made with sufficient accuracy by holding the spring stretched and testing it by a small level held above it. The length of the pendulum should be carefully adjusted so that the pendulum beats seconds as tested by counting the vibrations in two or three minutes. Until the pendulum is to be used for obtaining a record, it is held out of the vertical by a small lever. The silk thread should be of such a length that it becomes detached just before the pendulum reaches the floor. As the weight descends and the speed of the axis increases, the spring lengthens until the pins come opposite one another; this is the moment at which the thread should become detached. After a few trials the proper height from which to release the weight (as indicated by a vertical metre-stick) is readily ascertained.

Measurements. — When the preceding adjustments have been completed, the weight is allowed to descend. The brush is then inked, and as soon as the thread becomes detached the pendulum is released. After one vibration the pendulum is arrested so that the record may not be confused. The order in which the two lines on the paper were made can be ascertained from their slope. This record should be numbered in lead pencil and then several subsequent records may

be obtained in the same way on the same strip of paper without any confusion. When a sufficient number to give a good average have been obtained, the brush should be allowed to inscribe a complete circle on the paper, the pendulum remaining at rest. The paper may then be removed. A little thought will show how the speed of rotation may be deduced from the records.

If the spring has been calibrated, its tension can be deduced from its length and the calibration curve. If not yet calibrated, its length when stretched is carefully measured and it is then removed and calibrated. Or the following procedure may be adopted: attach a cord to the block so as to stretch the spring and allow the cord to hang over a pulley clamped to the framework, so that the part of the cord between the lead block and the pulley is horizontal and then place such weights in a pan carried by the cord that the pins come into line. While the apparatus is in this position, the radius of the path described by the centre of the block may be obtained by means of the beam-compass, measurements being made of both the inside and the outside distances of axis and block and the mean taken.

From the speed of rotation (in radians per second) and the radius of the circle described by the centre of the block, the acceleration toward the centre is calculated. From this and the mass of the block the central force is deduced and compared with the tension of the spring.

The same process should be repeated with a spring of different stiffness, again with a block of different mass, and again with a different radius of rotation (obtained by slipping the horizontal arm through the vertical axis).

Second Method. — An interesting variation of the above that gives good results and dispenses with recording disk, pendulum, and thread and weight may be briefly sketched. The axis is set into rotation by slight impulses from the thumb and forefinger on the lower end of the axis. By the same means it is kept in rotation at such a rate that when the moving pin passes the stationary one they are as nearly as possible in line. Only very slight impulses are needed, as the friction of the bearings is very slight. When the right speed of rotation has been obtained and can be kept up, the speed of rotation can be

found by counting the number of revolutions in a given time, say two or three minutes. If a stop-watch is used, this will present no difficulty. If a clock or chronometer circuit * is used for time, each minute may be regarded as beginning and ending at the first tick after a silence. Begin counting passages after the tick that indicates the beginning of a minute, call this the zero passage, and continue until the end of the two or three minutes. To obtain a good mean, this should be repeated several times.

DISCUSSION AND PROBLEMS

(*a*) Meaning and deduction of formula.

(*b*) Meaning of "centrifugal force."

(*c*) Force acting on vertical axis.

(*d*) Effect of spring not being horizontal.

(*e*) Effect of supporting cord not being vertical.

(*f*) How the mass of the rotating body could be deduced from this experiment.

(*g*) Direction and magnitude of whole resultant force on rotating body.

(*h*) At what inclination to the vertical would the supporting cord stand if the body rotated at the same speed but the spring were absent?

(*i*) What angular velocity must a boy give to a sling of 80 cm. length in order that the stone may not fall out when it is at the highest point?

(*j*) The centre of the moon is about 60 times the earth's radius from the centre of the earth, and it revolves once in 27 days 8 hours. Compare its acceleration with that of a body allowed to fall near the surface of the earth. Test the law of gravitation.

(*k*) State the formula for "centrifugal force" in gravitational units.

65. The Conical Pendulum. — A ball is attached to the end, P, of an arm, AP, that is pivoted at A to a vertical

* A circuit containing a relay or sounder and connected with a chronometer or clock in such a way that the relay sounds once per second, but fails to sound at the completion of each minute.

axis that rotates with an angular velocity ω. P revolves in a circle of radius r, and AP describes a cone of height h; hence P is acted on by a force $m\omega^2r$ directed toward C, the centre of the circle. The tension in AP may be resolved into a vertical component that supports the weight, mg, of the ball, and a horizontal component that supplies the force $m\omega^2r$ in the direction PC. Hence from the triangle PCA we get

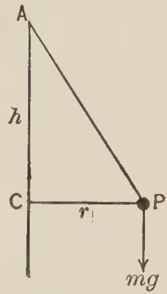

FIG. 35.

$$\frac{m\omega^2r}{mg}=\frac{r}{h}.$$

$$\therefore\ \omega=\sqrt{\frac{g}{h}}.$$

Watt's governor for a steam engine is essentially a double conical pendulum applied to the regulation of a steam valve.

Exercise XV. The Conical Pendulum

The same apparatus is used as in the preceding exercise, except that the cross-arm is removed and the short bar that carries the V-cord is placed in a hole in the vertical axis. The block with its vertical pin is adjusted until the end of the pin in the revolving ball just comes above the fixed pin, when the weight reaches the floor and the thread becomes detached from the axis. This adjustment having been made, the apparatus is brought to rest by a little pressure of thumb and forefinger on the vertical axis, the thread is rewound and the weight again allowed to descend from the same height, and when the weight reaches the floor the pendulum is released and a record of speed obtained. Or the second method of procedure of the preceding exercise (dispensing with the recording disk) may be employed.

The value of h cannot readily be found by direct measurement. Perhaps the best way is to measure AP and PC (Fig. 35), and deduce h. These distances must be measured from centre of ball to

centre of steel axis. Hence for each distance two measurements must be made with the beam-compass, an inside measurement and an outside measurement, the ball being meanwhile held by a cord that passes round it and is attached to the wooden support. The value of ω, calculated from h and g, should agree closely with the experimental value.

By changing the initial height from which the weight descends, different values of ω may be tried.

DISCUSSION

(*a*) Sources of error.

(*b*) What is the length of a simple pendulum that vibrates once during a revolution of the conical pendulum?

(*c*) What motion does the ball seem to have if viewed from a great distance in the plane of revolution?

(*d*) The vertical distance of a governor-ball below the pivot varies inversely as the square of the velocity of revolution.

(*e*) Calculate the tension of the suspension.

(*f*) If several pendulums of different lengths were attached to A, how would they hang when the rotation became steady?

(*g*) At what angle does a bicyclist tilt his bicycle in going around a curve?

(*h*) How strong must the spokes of a fly-wheel be to be able to stand all the strain without aid from the rim?

(*i*) How strong must the rim be to be able to stand all the strain without aid from the spokes?

66. Friction. — When the surfaces of two solids are in contact there is a resistance to sliding. This resistance is called *friction*. If a force tending to produce sliding be applied to one of the bodies, the other being kept at rest, sliding will not take place unless the force be above a certain value. For forces less than this critical value, the friction just equals the applied force and no sliding takes place. The force just necessary to produce sliding is a measure of the maximum *static friction* between the sur-

faces. So measured the maximum static friction is found to be proportional to the perpendicular pressure between the surfaces, at least throughout a considerable range of pressure. The ratio of the maximum static friction, F, to the perpendicular pressure, P, between the surfaces is called the *coefficient of static friction*, n, or

$$n = \frac{F}{P}.$$

When sliding has once begun it is found that a smaller force will suffice to continue the motion. The force that will just maintain the motion is a measure of the *kinetic friction* between the surfaces, and the ratio that it bears to the normal pressure is called the *coefficient of kinetic friction* between the surfaces. The general results of experiments on kinetic friction may be summarized as follows: (1) the ratio of the kinetic friction to the pressure, *i.e.* the *coefficient of kinetic friction*, is practically constant through a wide range of variation of pressure, (2) the ratio is also practically independent of the speed of sliding provided the latter be not very small, (3) when the speed is very small and decreases toward zero, the friction increases and approaches more and more the magnitude of the maximum static friction and at indefinitely small speeds the two are equal.

The coefficient of kinetic friction between surfaces of wood depends on the materials, varying between .25 and .50. In the case of metal surfaces it lies between .15 and .20.

67. Motion on an Inclined Plane. — A body on an inclined plane is acted on by two forces, gravity and friction. If

the mass of the body is m and the inclination of the plane to the horizontal is i, the weight, mg, of the body may be resolved into the component $mg \sin i$ parallel to the plane, and the component $mg \cos i$ perpendicular to the plane. If i is such that the body just begins to move when released, $mg \sin i$ is just equal to the maximum static friction; and since the pressure between the body and the plane is $mg \cos i$, the coefficient of static friction is

$$n = \frac{mg \sin i}{mg \cos i} = \tan i.$$

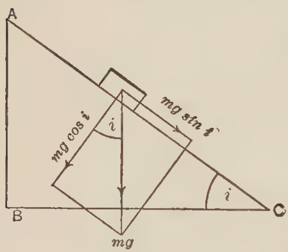

FIG. 36.

From this n may be determined.

If i is such that the body slides downward with an acceleration a, the whole force parallel to the plane must equal ma. This force consists of the component of gravity down the plane, $mg \sin i$, and the force of friction in the opposite direction, or $n' \cdot mg \cos i$, n' being the coefficient of kinetic friction. Hence

$$mg \sin i - n'mg \cos i = ma.$$

$$\therefore\ n' = \tan i - \frac{a}{g} \sec i.$$

Hence if i and a be measured, n' can be deduced. The value obtained for n' will, of course, be its value for the particular speed at which the body is moving when the acceleration is a.

Exercise XVI. Friction

(1) The coefficient of static friction of a pine block on a pine board is found by adjusting the latter to such an inclination that the block just slides when released. The same should be done with different

weights placed on the block to show how far the coefficient is independent of the pressure. The results of different trials will be more consistent the more uniform the surfaces are. Surfaces that have been freshly sandpapered and well brushed will give good results.

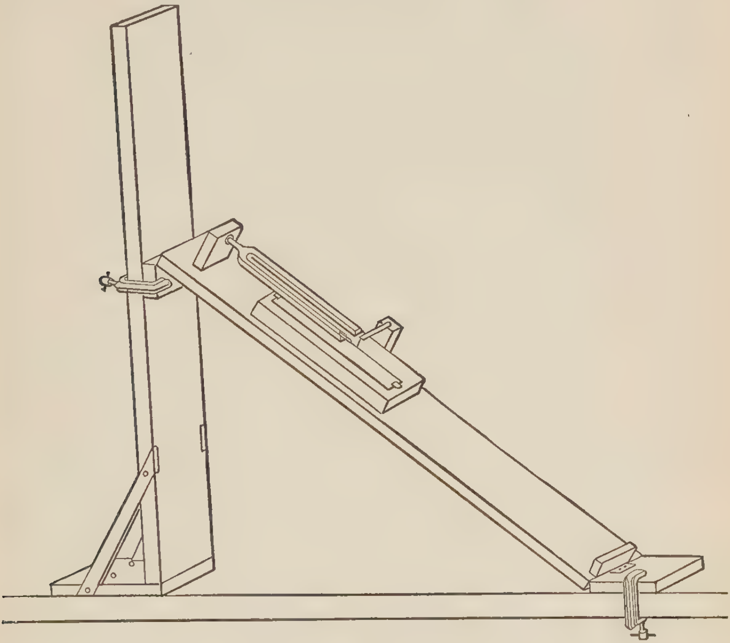

Fig. 37.

(2) To find the coefficient of kinetic friction of the same surfaces, adjust the board to a high inclination (about 60° to the horizontal), and attach a strip of glass coated with soap (bon ami) to the block by means of thumb-tacks. Place a tuning-fork with stylus (as in Exercise VI) so that the stylus will draw a curve on the glass as the block descends. A second stylus attached to the inclined plane should be adjusted so that, as the block descends, it will trace a (nearly straight) line that passes exactly through the middle of the waves traced by the first stylus. The two styli should be as close together

as practicable, and placed so that, when both are at rest, they trace but a single line on the glass as the block descends.

To find the acceleration of the block, measure, along the line traced by the fixed stylus, the length of several successive groups of four waves each. The measurement may be made by inverting the glass on a millimetre scale (the tape of Exercise V will do), and the readings should be as exact as possible. The time for each group is known from the frequency of the fork. Thus the mean velocity in each group is readily found, and a value of the acceleration may be deduced from each two successive groups. As many different values as possible should be obtained and averaged, and the coefficient of friction calculated from the average.

DISCUSSION

(*a*) How far did the results show that the coefficient of static friction is independent of the pressure?

(*b*) Did the results indicate any variation of coefficient of friction with velocity?

(*c*) Why did the straight line need to be drawn exactly through the middle of the wave line?

(*d*) If the block and board were horizontal, and the former were acted on by a horizontal force equal to its weight, with what acceleration would it move and how far would it go in 10 sec.?

(*e*) If the force in (*d*) act at the centre of the block at an angle α with the horizontal, what will α be if the block just start, (1) when the force is a pull; (2) when the force is a push?

(*f*) A body sliding down the length of a smooth plane attains the same velocity as if it fell vertically the height of the plane; but this is not so if the plane is rough.

68. Dimensions of Force and Momentum. — The unit of momentum is the momentum of a body of unit mass moving with unit velocity. It therefore varies directly as the unit of mass and also directly as the unit of velocity or $(mom) \propto (M)(V)$; but (§ 26) $(V) \propto (LT^{-1})$; hence $(mom) \propto (MLT^{-1})$. Instead of the sign of variation, $\propto$,

we may use the sign of equality, meaning *equality of dimensions*, and, as this is the more common method, we shall hereafter use it.

The unit of force is the force that gives unit of mass unit acceleration; it therefore varies directly as the unit of mass, and also directly as the unit of acceleration, or $(F) = (M)(A) = (MLT^{-2})$.

It is evident that these dimensional relations can be derived directly from equations connecting the quantities of unknown dimensions and other quantities of known dimensions. Thus from *momentum* $= ma$ we get $(mom) = (MLT^{-1})$, and from $F = ma$ we get $(F) = (MLT^{-2})$. In deriving dimensional relations by this method we neglect numerical constants, since they do not depend on the fundamental units or are of zero dimensions.

REFERENCES FOR CHAPTER V

Mach's "Science of Mechanics."

Macgregor's "Kinematics and Dynamics," Part II, Chapters I and II.

Lodge's "Pioneers of Science" (historical).

CHAPTER VI

MOMENT OF FORCE

69. In the preceding chapter we have considered the motion of translation produced by forces acting on a particle. When a force produces rotation of a body, the magnitude of the effect depends on something more than the magnitude and direction of the force and the magnitude of the mass moved. Every one knows that, to set a heavy wheel in rotation, the force should be applied as far from the axis of rotation as possible. The importance of a force as regards rotation, or the *moment* * of the force as it is called, depends on the magnitude and direction of the force, and also on its distance from the axis of rotation. The opposition offered by the inertia of the wheel is greater, the farther, on the whole, the mass of the wheel is from the axis of rotation. In other words, the importance of inertia as regards rotation, or the *moment* of inertia as it is called, depends on the distances of the parts of the body from the axis of rotation. In the following sections we shall arrive at more precise definitions of moment of force and moment of inertia.

70. Moment of Force and Moment of Inertia. — Consider a particle P, of mass m, free only to rotate about an

* The word *moment*, in the sense of *importance*, occurs in such phrases as "a matter of no moment."

axis, A, in a circle whose centre is C and radius r. Let a force F act on P in the plane of the circle C, then the only part of F that can affect the motion of P is the component tangential to the circle. Let the direction of F make an angle θ with the tangent to the circle; then the effective component of F equals $F\cos\theta$. If the linear acceleration of P be a,

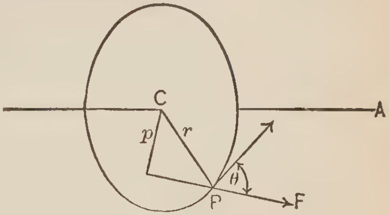

FIG. 38.

$$F\cos\theta = ma.$$

From C drop a perpendicular, p, on the line in which F acts. Then the angle between p and the radius through P is also θ and $\cos\theta = \frac{p}{r}$.

$$\therefore\ Fp = mr \cdot a.$$

Since this is a case of rotation only, it is more properly stated in terms of the angular acceleration α of the particle P. Now $a = \alpha r$

$$\therefore\ Fp = mr^2 \cdot \alpha.$$

Hence the effectiveness of the force F in producing rotation is measured by Fp. The product of the force F by its perpendicular distance from the axis is called the *moment of the force* about that axis.

If F be not perpendicular to the axis, it may be resolved into a component parallel to the axis and a component perpendicular to the axis. The only part of F that will tend to produce rotation about the axis will be the component perpendicular to the axis, and the moment of F about the

axis will be the component perpendicular to the axis multiplied by the distance of this component from the axis.

If, instead of being attached to the axis A, the particle m be entirely free, the components of F parallel to A and along CP will cause accelerations in those directions, but the result as regards rotation about A will not be changed.

The multiplier of a in the above equation, namely mr^2, depends on the mass of the particle and the distance from the axis. The product mr^2 is called the *moment of inertia* of the particle m about the axis.

71. Rotation of a Rigid Body. — As the simplest case of a rigid body, imagine two particles m_1 and m_2 in the plane of the paper connected together by a rod whose mass may be neglected, and suppose that they are only free to rotate about an axis through C perpendicular to the plane of the paper. Then at any moment the particles must have the same angular velocity and angular acceleration about C.

Let forces F_1 and F_2 act in the plane of the paper on the particles m_1 and m_2 respectively, and let perpendiculars from C on F_1 and F_2 be p_1 and p_2 respectively. Let the perpendicular from C on the connecting rod be p. Then since the stress T in the rod acts in opposite directions on the particles,

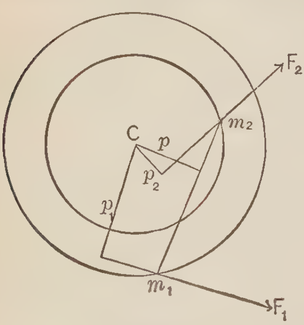

FIG. 39.

$$F_1p_1 + Tp = m_1r_1^2\alpha,$$

$$F_2p_2 - Tp = m_2r_2^2\alpha.$$

Hence,

$$F_1p_1 + F_2p_2 = (m_1r_1^2 + m_2r_2^2)\,\alpha.$$

Evidently we can extend the method here used to any number of particles rigidly connected together and therefore to a rigid body. Hence for a rigid body

$$\Sigma Fp = \alpha \cdot \Sigma mr^2.$$

Σmr^2, or the sum of the moments of inertia of the particles of the body about a particular axis, is called the moment of inertia of the body about that axis. It evidently depends only on the mass and form of the body.

ΣFp, or the sum of the moments of the various external forces about a particular axis, is the total moment of force about that axis. Denoting the moment of inertia of the body about a certain axis by I and the total moment of force about the axis by C,

$$C = I\alpha.$$

If the forces F_1 and F_2 do not act in the plane of rotation, the only parts of them we need consider are their components in the plane of rotation, and these are the only parts that contribute to the moment of force, C, about the axis of rotation. Hence the result is unchanged. If m_1 and m_2 rotate about the axis in two different planes perpendicular to the axis, the only difference in the above proof will be that the stress T will be inclined to the axis; but since its component in a plane perpendicular to the axis will still be a pair of equal and opposite forces acting on m_1 and m_2 respectively, the proof will still hold good. Hence the formula $C = I\alpha$ applies to a body of any shape. The analogy between the formula for rotation and the formula, $F = Ma$, for translation should be noted.

72. Moment of Inertia of a Uniform Rod. — Let the length of the rod be L and its mass M. We shall suppose that

the rod is of constant cross-section and that its thickness is small compared with its length.* Let the length, L, of the rod be divided up into a large number, N, of short equal parts each of mass $\frac{M}{N}$ and length $\frac{L}{N}$. The nth part reckoning from one end is at a distance $\frac{nL}{N}$ from that end, and its moment of inertia about that end is $\left(\frac{M}{N}\right)\left(\frac{nL}{N}\right)^2$. Hence, summing up for all values of n from 1 to N, the total moment of inertia is

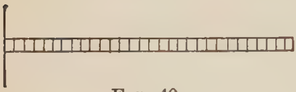

FIG. 40.

$$\Sigma\left(\frac{M}{N}\right)\left(\frac{nL}{N}\right)^2=\left(\frac{ML^2}{N^3}\right)\Sigma n^2,$$

$$=\left(\frac{ML^2}{N^3}\right)\frac{1}{6}N(N+1)(2N+1).$$

$$=ML^2\,\frac{1}{6}\left(1+\frac{1}{N}\right)\left(2+\frac{1}{N}\right).$$

If N be supposed indefinitely large, $\frac{1}{N}=0$.

$$\therefore\ I=\tfrac{1}{3}ML^2.$$

To deduce the moment of inertia of a rod about its centre we have only to apply this formula to the two halves of the rod.

$$I=\frac{1}{3}\left(\frac{M}{2}\right)\left(\frac{L}{2}\right)^2+\frac{1}{3}\left(\frac{M}{2}\right)\left(\frac{L}{2}\right)^2,$$

$$=\tfrac{1}{12}ML^2.$$

* The student who is familiar with the differential and integral calculus may substitute the following:

Let ρ be the mass of unit length of the rod. The mass of a short length dr is ρdr and its moment of inertia about one end of the rod is $\rho r^2 dr$.

$$\therefore\ I=\int_0^L \rho r^2 dr=\tfrac{1}{3}\rho L^3=\tfrac{1}{3}ML^2.$$

73. Moment of Inertia of a Uniform Rectangular Disk. — (1) *About an axis in the plane of the disk and bisecting the sides whose length is a.* Suppose the whole rectangle divided up into narrow strips parallel to the sides a. Applying to these the formula for the moment of inertia of a rod and adding for all the strips, we get

$$I_1 = \tfrac{1}{12} Ma^2.$$

(2) *About an axis in the plane of the disk and bisecting the sides whose length is b.* $I_2 = \tfrac{1}{12} Mb^2.$

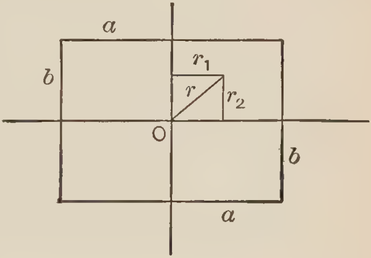

FIG. 41.

(3) *About an axis through the centre perpendicular to the disk.* The moment of inertia of a particle, m, whose distance from the centre is r is mr^2. If r_1 and r_2 be the distances of m from the axes considered in (1) and (2),

$$r^2 = r_1^2 + r_2^2.$$

$$\therefore\ mr^2 = mr_1^2 + mr_2^2.$$

Hence if we sum up both sides of the equation for all particles in the disk, we get

$$I = I_1 + I_2.$$
$$= \tfrac{1}{12} M(a^2 + b^2).$$

NOTE. — It is obvious that the method here used for finding the moment of inertia about an axis perpendicular to the plane of the disk applies to a disk of any form. If its moment of inertia about any two rectangular axes in the plane of the disk be I_1 and I_2, then its moment of inertia about a third axis passing through the intersection of the first two and perpendicular to the planes of the disk is $I = I_1 + I_2$.

74. Moment of Inertia of a Rectangular Block. — The block may be divided up into disks parallel to one face. The moment of inertia of each disk about an axis, through its centre and perpendicular to its plane, is given by (3) of the preceding section. Hence, adding for all the disks we get

$$I = \tfrac{1}{12} M(a^2 + b^2),$$

M being the mass of the block, and a and b the sides of the face to which the axis is perpendicular.

75. Moment of Inertia of a Uniform Circular Disk. — Let the radius of the disk be R, and its mass M.* Suppose the disk divided up into a large number, N, of concentric rings, each of width $\frac{R}{N}$. The area of the nth ring from the centre is $2\pi n \frac{R}{N} \cdot \frac{R}{N} = 2\frac{n}{N^2} \cdot \pi R^2$. Now πR^2 is the area of the whole disk. Hence the area of the ring is the fraction $\frac{2n}{N^2}$ of the area of the whole disk. Hence, M being the mass of the disk, the mass of the ring is $\frac{2n}{N^2} M$. The moment of inertia of

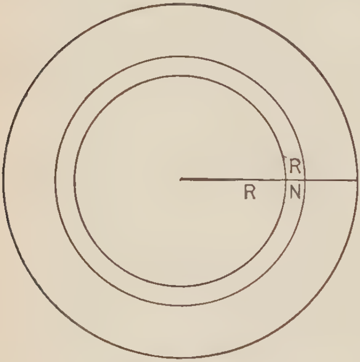

FIG. 42.

* *By calculus method.* Let ρ be mass of the disk per unit area. The mass of a narrow ring of radius r, and width dr, is $2\pi r \cdot dr \cdot \rho$, and its moment of inertia about the axis of the disk is $2\pi r\, dr\, \rho \cdot r^2$.

$$\therefore\ I = \int_0^R 2\pi\rho r^3 dr = \tfrac{1}{2}\pi\rho R^4 = \tfrac{1}{2} MR^2.$$

this ring about an axis through the centre perpendicular to the plane of the disk is $\frac{2n}{N^2}M \cdot \left(n\frac{R}{N}\right)^2$. Summing up for all values of n from 1 to N, we get

$$I = \frac{2MR^2}{N^4}\Sigma n^3$$

$$= \frac{2MR^2}{4}\left\{\frac{N(N+1)}{2}\right\}^2$$

$$= 2MR^2\left\{\frac{1\left(1+\frac{1}{N}\right)}{2}\right\}^2.$$

Hence, supposing N indefinitely great,

$$I = \tfrac{1}{2}MR^2.$$

It is readily seen from the note in § 73 that the moment of inertia of the circular disk about a diameter is $\frac{1}{4}MR^2$.

76. Moment of Inertia of a Right Circular Cylinder. — A right circular cylinder may be divided up into circular disks. The preceding formula applies to each disk. Adding the moments of inertia of all the disks, we get for the moment of inertia of a right circular cylinder of mass M about its geometrical axis

$$I = \tfrac{1}{2}MR^2.$$

It will be shown later that the moment of inertia of a right circular cylinder about an axis parallel to its geometrical axis, and at a distance d from the latter, is

$$I = \tfrac{1}{2}MR^2 + Md^2.$$

77. Radius of Gyration. — The radius of gyration, k, of a body about a certain axis is a length such that, if the

whole mass were supposed concentrated at that distance from the axis, the moment of inertia would be unchanged, or $I = Mk^2$. For a rod about the middle, $k^2 = \frac{1}{12} L^2$. For a circular disk about the axis of figure $k^2 = \frac{1}{2} R^2$, and so on.

78. Angular Momentum. — If a body have an angular velocity, ω, and a moment of inertia, I, about any axis, the product $I\omega$ is called the *angular momentum* about that axis. Supposing I to remain constant, the rate of change of $I\omega$ is I multiplied by the rate of change of ω, *i.e.* $I\alpha$. But $C = I\alpha$, hence the moment of force about an axis equals the rate of change of angular momentum about that axis. If the moment of force is zero, the angular momentum is constant.

Exercise XVII. Moment of Force and Angular Acceleration

Apparatus. — A horizontal circular disk of wood is carried by a vertical steel axis which passes through the centre of the disk. The steel axis is mounted in cone bearings, the bearing points being steel needle-points so that the friction is reduced to a minimum. A silk thread is wrapped around the axis and passing over a pulley carries a weight the position of which can be observed by a vertical scale. Lead cylinders are placed on the disk at equal distances on opposite sides of the axis so as to increase the moment of inertia. The mass of each part of the apparatus is stamped on it. The diameter of the steel axis may be measured by a micrometer caliper (§ 3). A metre scale and a simple beam-compass (§ 3) are used for measuring the diameter of the disk, the diameter of the lead cylinders and the distance of the latter from the axis.

Calculation of Time of Descent of Weight. — From the dimensions and masses of the various parts their moments of inertia about the axis of rotation are calculated. The moment of the force exerted by the silk thread on the axis is deduced from the tension of the thread,

and the radius of the axis. The angular acceleration is derived from the total moment of inertia and the moment of the force. The time required for the weight to descend to the floor is then calculated.

Observation of Time of Descent of Weight. — The disk is levelled by means of a spirit level. To prevent tangling the length of the thread

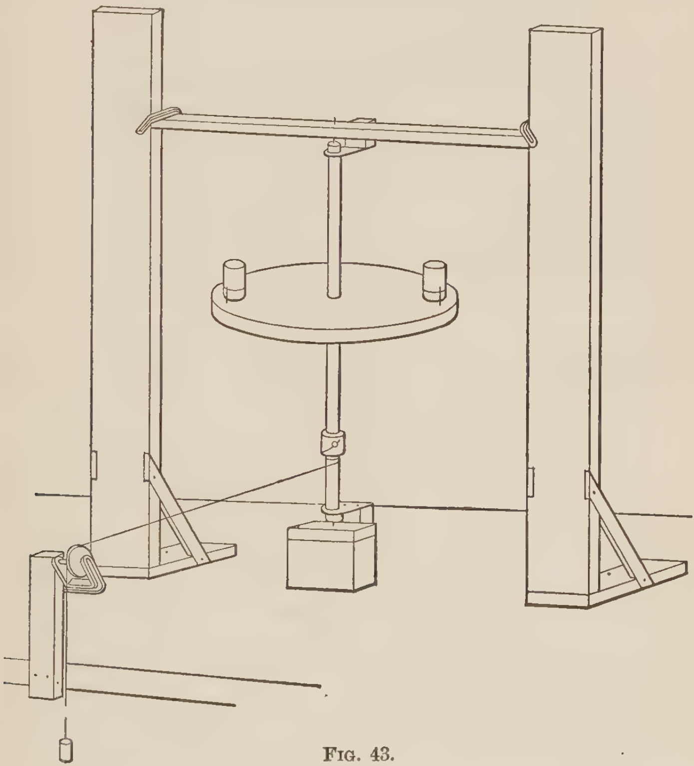

FIG. 43.

should be such that the weight will just reach the floor. To prevent overlapping of the windings of the thread, it should be attached to the axis at a point slightly higher or lower than the point at which it tends to wind when the disk is turned. With this adjustment the thread will always be very nearly at right angles to the axis. The

upper bearing should be adjusted until the axis is just perceptibly loose in the bearings.

The disk is then rotated and the thread wound on the axis without overlapping until the weight rises to the height desired. Then exactly at a tick of the clock the disk is released and the number of whole seconds and fifths of a second before the weight strikes the floor is carefully noted. This should be repeated ten times and a strong effort made to have the separate determinations as independent as possible. The average of these is taken as the time of descent.

(The changes of energy that take place will be studied in Exercise XXIV.)

DISCUSSION

(*a*) Meaning and proof of formulæ used.

(*b*) Is it necessary that the axis be vertical or the thread horizontal or is it sufficient that they be at right angles?

(*c*) Effect of using a thread whose thickness is comparable with the radius of the axis.

(*d*) Effect of stretching of thread and overlapping of thread.

(*e*) Was the tension just equal to the weight? How could it be found exactly?

(*f*) Effect of bearings not being quite central.

(*g*) Is there any reason to suppose that the acceleration is not quite constant?

(*h*) If, after the weight has reached the floor, the disk be allowed to continue in rotation and rewind the thread, why will not the weight rise to its original height?

(*i*) What would be the effect if the cylinders were made of the same mass as before but of twice as great diameter?

(*j*) An iron cylinder, 3 ft. in external diameter and 2 ft. 10 in. in internal diameter, rolls down a plane 20 ft. long inclined at 30° to the horizontal. What linear velocity does it acquire?

79. Centre of Mass of a Body.* — The centre of mass (or of inertia) is a point of great importance in the study of

* The part of this section preceding the definition in italics may (if thought advisable) be omitted. The only objection to this is that it will involve the assumption that, if a point fulfil the condition of being the

a group of particles or of a rigid body. The centre of mass of a particle m_1 at P_1, and a particle m_2 at P_2, is a point Q_1 in P_1P_2 such that

$$m_1 \cdot Q_1P_1 = m_2 \cdot Q_1P_2,$$

or
$$m_1 : m_2 :: Q_1P_2 : Q_1P_1.$$

If a third particle m_3 at P_3 be added, the centre of mass of all three is a point Q_2 in Q_1P_3 such that

$$m_1 + m_2 : m_3 :: Q_2P_3 : Q_2Q_1,$$

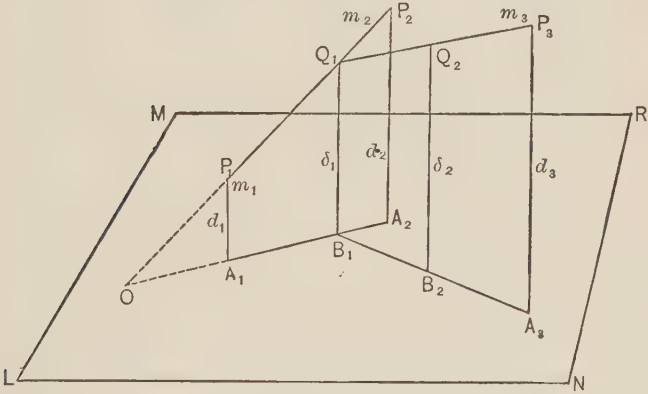

FIG. 44.

and so on for any number of particles, and hence for a rigid body.

The centre of mass can be more briefly defined in terms of the distances of the particles from any plane. Let the perpendiculars from P_1, P_2, P_3, $\cdots$, Q_1, Q_2, $\cdots$, on any plane, *LMNR*, be respectively d_1, d_2, d_3, $\cdots$, ∂_1, ∂_2, $\cdots$, and

centre of mass as regards distances from any three planes, it will also fulfil the condition as regards distances from any other plane. The proof of this proposition would require more knowledge of analytical geometry than we assume in this book.

let the feet of the perpendiculars be A_1, A_2, A_3, $\cdots$, B_1, B_2, $\cdots$, respectively. Produce P_1P_2 and A_1A_2 to meet in O. Then

$$Q_1P_2 : Q_1P_1 :: OP_2 - OQ_1 : OQ_1 - OP_1.$$
$$:: P_2A_2 - Q_1B_1 : Q_1B_1 - P_1A_1.$$
$$\therefore\ m_1 : m_2 :: d_2 - \partial_1 : \partial_1 - d_1,$$

or
$$(m_1 + m_2)\partial_1 = m_1 d_1 + m_2 d_2.$$

It can be shown in the same way that

$$Q_2P_3 : Q_2Q_1 :: P_3A_3 - Q_2B_2 : Q_2B_2 - Q_1B_1.$$
$$m_1 + m_2 : m_3 :: d_3 - \partial_2 : \partial_2 - \partial_1.$$

Hence
$$(m_1 + m_2 + m_3)\partial_2 = (m_1 + m_2)\partial_1 + m_3 d_3,$$
$$= m_1 d_1 + m_2 d_2 + m_3 d_3.$$

The process can evidently be extended to any number of particles, and so to a continuous body. Hence we may define centre of mass thus: *If* m_1, m_2, $\cdots$ *are the respective masses of the particles constituting a body (or group of particles) of total mass* M, *and if the respective distances of these particles from any plane are* d_1, d_2 $\cdots$, *the centre of mass is a point whose distance from the plane is*

$$\partial = \frac{m_1 d_1 + m_2 d_2 + \cdots}{m_1 + m_2 + \cdots} = \frac{\Sigma md}{M}.$$

If m_1, m_2 $\cdots$ are all equal, ∂ is the mean of the distances d_1, d_2 $\cdots$. If m_1, m_2 $\cdots$ are unequal, ∂ is still, in a sense, the mean distance, but, in taking the mean, the distance of each particle is given an "importance" measured by the mass of that particle. In applying the definition to a plane that passes through the group of particles or body,

distances on one side of the plane must be considered positive and those on the other side negative.

If d_1, d_2, etc., be the distances of the particles m_1, m_2, etc., from a plane that passes through the centre of mass, $\partial = 0$ and therefore $\Sigma md = 0$.

80. Coördinates of Centre of Mass. — The position of the centre of mass can be specified definitely by its distances (with proper signs) from any three planes at right angles. If in any problem three intersecting lines at right angles have been chosen as axes of rectangular coördinates, then the distance of any particle m from the plane containing the x and y axes is z, its distance from the plane of the y and z axes is x, and its distance from the plane of the x and z axes is y. Hence, if $\bar{x}$, $\bar{y}$, $\bar{z}$ be the coördinates of the centre of mass,

$$\bar{x} = \frac{\Sigma mx}{M}; \quad \bar{y} = \frac{\Sigma my}{M}; \quad \bar{z} = \frac{\Sigma mz}{M}.$$

81. Centre of Mass in Simple Cases. — When a homogeneous body has a geometrical centre, the body can be supposed divided up into pairs of equal particles, each pair lying in a line through the geometrical centre and one of the pair being as far on one side of the centre as the other is on the other side. Hence the geometrical centre is also the centre of mass. Hence the centre of mass of a uniform rod, a uniform circular disk, a sphere, a right circular cylinder, a parallelogram, a parallelepiped, etc., is in each case the geometrical centre.

When a body can be divided up into a number of parts the masses and centres of mass of which are known, the

centre of mass of the whole body can be found by means of the expressions for $\bar{x}$, $\bar{y}$, $\bar{z}$ in § 80.

82. Moment of Inertia of a Body about any Axis. — Let I be the moment of inertia of a body about an axis through A perpendicular to the plane of the paper and I_0 its moment of inertia about a parallel axis through the centre of mass C. Let m be a particle at a point P. From P draw perpendiculars PA and PC to the two axes. Join AC and from P draw a perpendicular PD on AC. Denote CD by d. Then

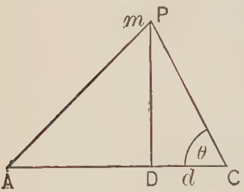

FIG. 45.

$$
\begin{aligned}
I &= \Sigma m PA^2 \\
&= \Sigma m(PC^2 + CA^2 - 2\,PC \cdot CA \cos PCA) \\
&= \Sigma m PC^2 + CA^2 \Sigma m - 2\,CA \cdot \Sigma(m PC \cos PCA).
\end{aligned}
$$

But $\Sigma(mPC \cos PCA) = \Sigma md = 0$ since d is the distance of m from a plane through the centre of mass C and perpendicular to AC (§ 79). Hence if we denote the distance AC between the two axes by a and the mass of the body by M

$$I = I_0 + Ma^2.$$

As an example of the usefulness of this proposition consider the moment of inertia of a circular cylinder of mass M, length L, and radius R about an axis through the centre of the cylinder and perpendicular to the geometrical axis. Divide the whole length of the cylinder into a large number of equal disks by planes perpendicular to the axis of the cylinder. The moment of inertia of a disk of mass m at a distance x from the centre of the cylinder is (§ 75)

$$\tfrac{1}{4} mR^2 + mx^2.$$

The sum of the first term for all the disks is $\frac{1}{4} MR^2$. The summation of the second term has been performed already in finding the moment of inertia of a rod about its centre (§ 72). Hence

$$I = M\left(\frac{L^2}{12} + \frac{R^2}{4}\right).$$

83. The Conservation of Angular Momentum. — The angular momentum of a body about an axis is constant if the moment of force about that axis is zero (§ 78). A similar statement may be made with regard to the total angular momentum of a group of bodies (*e.g.* the solar system) about an axis or line in space, even if the bodies have different angular velocities and are not at fixed distances from the axis. The moment of each force about the axis is equal to the angular momentum it produces per second, and if the sum of the moments of all the forces about the axis is zero the total change of angular momentum in any time is zero. Internal forces, that is forces which the bodies exert on one another, are made up of pairs of equal and opposite forces (Newton's Third Law), and have therefore zero total moment about any axis. If the external forces that act on the system have also zero total moment about the axis, the angular momentum about the axis will be constant. This principle is called the *conservation of angular momentum*. The following exercise will illustrate a particular case.

Exercise XVIII. Conservation of Angular Momentum

A number of bodies, rotating about an axis with a known angular velocity, move to greater distances from the axis; find by calculation and also by experiment the new angular velocity.

A vertical steel axis supported on needle-points carries a horizontal steel cross-bar, along which two cylindrical iron blocks can slide. The axis is set into rotation by a thread that is wrapped

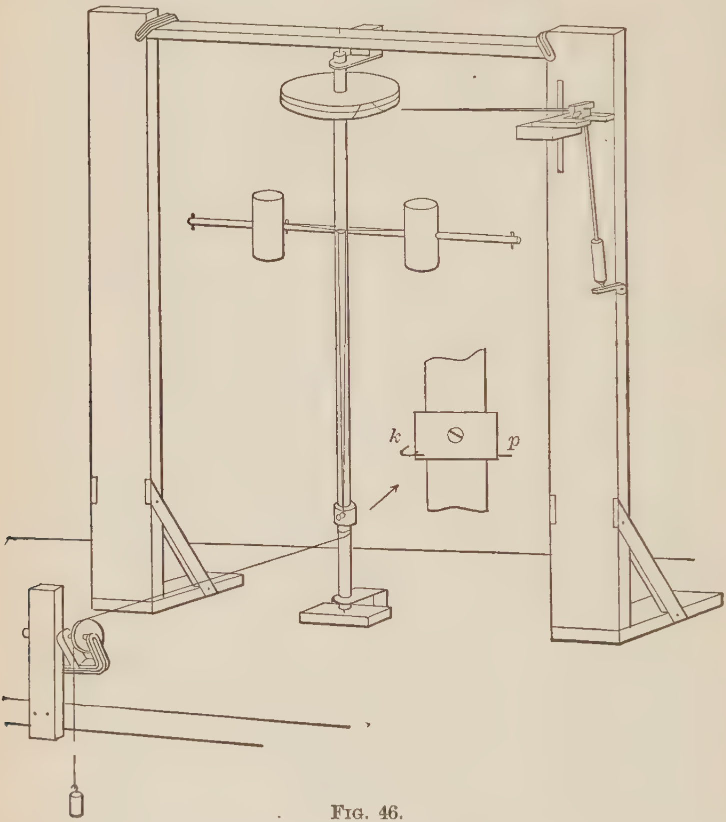

FIG. 46.

around the axis, and, passing over a pulley, carries a weight. By a simple device, a cord that restrains the sliding blocks is released at the moment when the thread is wholly unwrapped from the axis, or, by a slight change in the arrangement, the thread can unwrap without releasing the weights. (The cord and the thread loop over

a small peg, p; if the cord be outside it will be released; if inside, it will not be released. If the cord and thread be placed on the hook, k, the thread will not be detached from the axis, and the cord will be released or not released, according to whether it is outside or inside.) The initial positions of the blocks are fixed by pins in the cross-bar, and when the blocks slide out they are arrested in definite positions by other pins in the cross-bar. The speed of rotation of the axis at any time is ascertained by means of the recording disk, pendulum, and brush described in Exercise XIV (p. 84). Weigh blocks, rods, and disk before setting the apparatus up.

Several records of the final speed should be obtained, the apparatus being so arranged that the blocks are not released. For each record the cord is wrapped up anew, and the weight is allowed to descend from the same height. Again, several records should be obtained after the blocks have been released. For these experiments the thread should be placed on the peg, p, so that it will become detached and not retard the motion to be measured. Each record should be numbered as soon as obtained.

The distances of the centres of the blocks from the centre of the axis in both positions of the blocks should be measured very carefully by means of the beam-compass. The length of the cross-bar, the diameter of the disk, and the diameter of the axis should also be measured, the last by means of the micrometer caliper. From these measurements the total moment of inertia, both before and after the change of position of the blocks, can be calculated. Taking the initial angular velocity as known, calculate the final angular velocity.

With a view to facilitating Exercise XXVII, which is a continuation of the above, record the total distance of descent of the weight, the height to which it reascends when the thread is not detached and the moment of inertia is not changed, and the height to which it reascends when the moment of inertia changes.

DISCUSSION

(*a*) Sources of error.

(*b*) Meaning and proof of formulæ used.

(*c*) "Centrifugal force" of the blocks in both positions.

(*d*) At what speed of rotation (given the coefficient of friction) would the blocks just begin to slide if released?

(*e*) Does the friction between block and cross-bar affect the result?

(*f*) How could data be obtained for drawing a curve to show the way in which the speed increases with distance of descent of the weight, and how could a more accurate value of the final speed be thus obtained?

(*g*) Is the motion of the blocks affected in any way by force applied to them by the cross-bar?

(*h*) Why does water in a wash-basin rotate as it runs out?

(*i*) How must shrinkage of the earth, due to cooling, affect the length of the day?

84. Velocity and Acceleration of Centre of Mass. — The velocity of a particle in any direction is the rate of change of its distance from a fixed plane perpendicular to that direction. Let the distance of the particles $m_1, m_2, \cdots$ from a plane be $d_1, d_2, \cdots$ respectively, and let the distance of the centre of mass from the plane be δ, then (§ 79)

$$(m_1 + m_2 + \cdots)\delta = m_1 d_1 + m_2 d_2 + \cdots.$$

After a short interval of time, τ, let these distances be

$$\delta', d_1', d_2' \cdots;$$

then $$(m_1 + m_2 + \cdots)\delta' = m_1 d_1' + m_2 d_2' + \cdots.$$

If the first equation be subtracted from the second, and both sides of the result divided by τ, and τ then supposed indefinitely short, we get

$$(m_1 + m_2 + \cdots)\bar{v} = m_1 v_1 + m_2 v_2 + \cdots,$$

$\bar{v}$ being the velocity of the centre of mass in the direction in which the distances are measured and $v_1, v_2, \cdots$ the velocities of $m_1, m_2, \cdots$ respectively, in the same direction.

Applying the same method to changes of velocity we

can show that if $\bar{a}$ be the acceleration of the centre of mass in any direction, and a_1, a_2, $\cdots$ the accelerations of the particles m_1, m_2, $\cdots$ respectively,

$$(m_1 + m_2 + \cdots)\bar{a} = m_1a_1 + m_2a_2 + \cdots.$$

These two equations may also be written in the forms

$$\Sigma m(v - \bar{v}) = 0 \text{ and } \Sigma m(a - \bar{a}) = 0.$$

Here $(v - \bar{v})$ for any particle m is its velocity, in any direction, relatively to the centre of mass, and a similar statement applies to $(a - \bar{a})$.

85. Acceleration of the Centre of Mass of a Body acted on by External Forces. — The forces acting on a group of particles may be classified as internal and external. Internal forces are due to the actions and reactions between particles themselves. External forces are the forces between the particles and outside bodies. The internal forces on a bridge are the stresses in the various parts, the external forces are gravity, wind pressure, reaction of supports, etc.

Let the component, in any given direction, of the resultant of the external forces on a particle m_1 be F_1 and let the component in the same direction of the resultant of the internal forces on the particle be f_1; for a second particle m_2 let the corresponding components be F_2 and f_2 and so on. If the accelerations in that direction of the various particles be a_1, a_2, $\cdots$ respectively, then by Newton's Second Law

$$F_1 + f_1 = m_1a_1,$$

$$F_2 + f_2 = m_2a_2, \text{ etc.}$$

$$\therefore \Sigma F + \Sigma f = \Sigma ma.$$

If a be the acceleration of the centre of mass and M the whole mass, then by § 84

$$M\bar{a} = \Sigma ma.$$

By Newton's Third Law the internal forces occur in equal and opposite pairs,

$$\therefore\ \Sigma f = 0.$$

Hence $$\Sigma F = M\bar{a}.$$

Thus *the motion of the centre of mass in any direction is the same as if all the mass were concentrated at the centre of mass and all forces were transferred, with their directions unchanged, to the centre of mass.* Since this statement applies to any group of particles, it applies also to a continuous body.

The motion of the centre of mass of a body is not affected by internal forces. When a rocket explodes, the position and motion of the centre of mass are not affected by the explosion. Similar statements may be made with reference to attracting and colliding bodies.

86. Translation and Rotation. — If the linear motion of the centre of mass C of a body and the angular motion of the body about C are known, the whole motion of the body is known. To learn how any number of forces applied to the body affect its motion, it is only necessary to find the linear acceleration they impart to C and the angular acceleration about C which they produce. Each of these may be calculated separately. For the former depends only on the magnitudes and directions of the forces and not on their points of application, that is, on their moments about C. The latter depends only on the

moments of the forces about axes through C and is unchanged if additional forces be applied to C to keep it at rest, for such forces would have no moments about C. Thus we may consider translation of the centre of mass and angular acceleration about the centre of mass as the two independent effects of the forces applied to a body.

87. D'Alembert's Principle. — The equation proved in § 85 is very important. Written in the form $\Sigma F - \Sigma ma = 0$, it is sometimes called *D'Alembert's Principle*. Since ma equals the force that would give the mass m the acceleration a, it was sometimes called the "effective force" acting on the particle and $-ma$ was called the "reversed effective force." ΣF is the sum in a certain direction of the components of the external forces, and $-\Sigma ma$ is the sum in the same direction of the components of the "reversed effective forces," and since these sums of components added together equal zero, the whole of the external forces and the whole of the reversed effective forces may be considered as a system in equilibrium. Hence D'Alembert's Principle is usually stated thus: "The external forces with the reversed effective forces of a system of particles constitute together a system of forces in equilibrium."

Exercise XIX. Friction. D'Alembert's Principle

Four rectangular blocks of the same wood with plane, clean, freshly sandpapered surfaces are placed on one another as in Fig. 47, the lower one resting on another block of like material and finish. What force, F, applied horizontally to one of them, for example the third from the top, will pull it free of the others?

Let the masses be M_1, M_2, M_3, M_4 respectively and the respective accelerations a_1, a_2, a_3, a_4. The reader will find little difficulty in show-

ing that a_4 must be zero. The force F must be such as to produce in M_3 an acceleration, a_3, greater than the acceleration, a_2, of M_2. If P be the pressure between two of the blocks and μ the coefficient of static friction, the greatest horizontal force one can exert on the other is $P\mu$. Hence the greatest force possible between M_1 and M_2 is $M_1 g\mu$. If no slipping takes place between M_1 and M_2, the force between them will be less than $M_1 g\mu$, say $M_1 g\mu - \delta$, δ being either zero or a positive quantity.

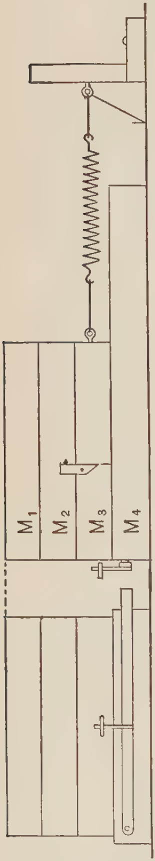

Fig. 47.

We can by D'Alembert's Principle write down the equations for the following systems: (1) M_1 alone, (2) M_1 and M_2 together, (3) M_1, M_2, and M_3 together.

$$M_1 g\mu - \delta - M_1 a_1 = 0. \qquad (1)$$

$$(M_1 + M_2) g\mu - (M_1 a_1 + M_2 a_2) = 0. \qquad (2)$$

$$F - (M_1 + M_2 + M_3) g\mu - (M_1 a_1 + M_2 a_2 + M_3 a_3) = 0. \qquad (3)$$

Subtracting (1) from (2), we get

$$M_2 g\mu + \delta - M_2 a_2 = 0.$$

Hence $a_2 \geqq g\mu$. Hence when slipping between M_2 and M_3 just takes place, $a_3 \geqq g\mu$.

Subtracting (2) from (3), we get

$$F - (2\,M_1 + 2\,M_2 + M_3)\,g\mu - M_3 a_3 = 0.$$

$$\therefore\ F \geqq 2\,(M_1 + M_2 + M_3)\,g\mu.$$

The student should also solve the problem by writing down Newton's Second Law for each one of the blocks. This will make it clear to him that D'Alembert's Principle is equivalent to Newton's Laws of Motion.

To find the least force, F, that will cause slipping between M_2 and M_3, attach a calibrated spring to M_3 and fasten the other end of the spring to an upright by means of a string of adjustable length (a bracket screwed to the table will do for the upright, and the string may be passed through a hook in the bracket). The spring should be horizontal; it should be attached to the middle of the end of M_3, and

its direction should be perpendicular to the end of M_3. Several trials will be necessary in order to find the length to which the spring must be stretched so that when M_3 is released, slipping takes place between it and M_2.

The following two simple devices will greatly facilitate the observations. A lever attached to the end of M_4 and carrying a stud that passes through a screw-eye in the end of M_3 is convenient for releasing M_3 promptly and uniformly. A small paper index fastened by a pin to the side of M_3 and pressing against a pin on the side of M_2 will be displaced if any slipping between M_2 and M_3 takes place, and is convenient for recording any actual slip. Without it a slight backward slip of M_2 at start of M_3, being compensated by an equal forward slip when M_3 stops, would sometimes pass unnoticed. When the desired adjustment of the length of the spring has been obtained, the length of the spring must be measured with great accuracy by means of the beam-compass or mirror-scale.

The coefficient of friction between M_3 and M_4 is found from the force (applied by the spring) necessary to cause M_3 to move on M_4. If the surfaces of the blocks have been freshly sandpapered, the coefficients of friction between the blocks will be found appreciably equal. To obtain the value of the coefficient as accurately as possible, find the friction between M_3 and M_4 when all four blocks are in position, then when M_1 has been removed, and finally when M_1 and M_2 have been removed. By plotting the various values of the friction against the corresponding values of the pressure of M_3 on M_4, a very reliable estimate of the coefficient of friction should be obtained and at the same time the constancy of the ratio of friction to pressure will be tested. Other arrangements of these positions of the blocks may be tried if time permit. (Instead of a spring in the preceding exercise, a cord that passes over a pulley and carries a pan and weights might be used. This would complicate the calculation, owing to the inertia of the pan and weights and the friction of the pulley, and on the whole gives less satisfactory results.)

DISCUSSION

(*a*) If sliding of M_2 were prevented, what force would just start M_3?

(*b*) What force would just cause M_3 to slide on M_4?

(*c*) Try to state in general language why the force required to pull M_3 free from M_2 and M_4, as in the exercise, is greater than that calculated in (*a*).

(*d*) Why does not M_4 move?

(*e*) Has M_1 any tendency to slide on M_2 in the experiment and why?

(*f*) In what respect would the solution of the problem of the exercise be different if the coefficients of friction between the various pairs of blocks were different?

(*g*) At what angle to the horizontal would the platform on which M_4 rests have to be tilted so that M_4 would slide downward?

(*h*) If the platform were tilted as in (*g*), between what pair of blocks would sliding first begin?

(*i*) If the platform were tilted, but not sufficiently to produce sliding, what amount of force would be called into play between each pair of blocks?

REFERENCES

Gray's "Treatise on Physics," Vol. I, Chapter IV.

Macgregor's "Kinematics and Dynamics," Part II, Chapters V and VI.

CHAPTER VII

RESULTANT OF FORCES. EQUILIBRIUM

88. Resultant of Forces acting on a Body. — The forces acting on a *particle* may be reduced to a single equivalent force called the resultant of the forces. When forces act at different points of a *body*, they may in certain cases be reduced to a single equivalent force. In other cases no single force will produce the same results as the actual forces. For example, suppose the centre of mass of a body has a linear acceleration in a certain direction, while the body has an angular acceleration about an axis in that direction through the centre of mass. A little consideration will show that no single force could produce exactly the same linear and angular accelerations.

The *resultant of the forces* acting on a body is the single force or the simplest set of forces that will give the body the same accelerations, linear and angular, as the actual forces produce. Hence we get the following conditions that the resultant must satisfy:

(1) The resultant must give the *centre of mass* of the body the same *linear acceleration* in any direction as the actual forces produce. Hence *the component of the resultant in any direction must equal the sum of the components of the actual forces* in that direction. It is, however, not necessary to consider all directions through the centre of mass. For an acceleration, a, in any direction is equivalent to

three accelerations, a_1, a_2, a_3, in directions at right angles, such that
$$a^2 = a_1^2 + a_2^2 + a_3^2.$$
A force that will cause these component accelerations will give rise to a and will satisfy this condition. Hence it is sufficient to consider three directions at right angles.

(2) The resultant must give the body the same *angular acceleration* about any axis as the actual forces produce. Hence *the moment of the resultant about any axis must equal the sum of the moments of the actual forces* about that axis. For a reason precisely similar to that stated in (1), it is only necessary to consider three rectangular axes through a point. Moreover, it is not actually necessary to consider rectangular axes through all points. For (§ 86) the two independent motions of a body are linear motion of the centre of mass and angular motion about an axis through the centre of mass. Hence it is sufficient to consider rectangular axes through the centre of mass only.

89. Resultant of Two Parallel Forces. — (1) Let the forces P and Q be in the same direction and let their points of application be A and B respectively. Consider a single force $R = P + Q$ parallel to P and Q and applied at a point C in AB such that $P \cdot AC = Q \cdot BC$. We shall show that the force R satisfies the conditions for being the re-

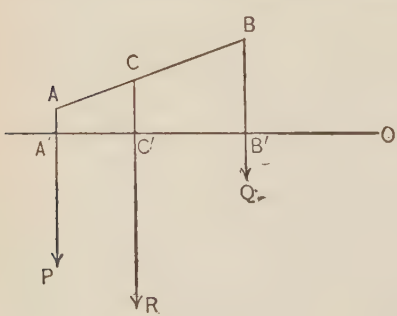

FIG. 48.

sultant of P and Q. Since R is in the same direction as P and Q and equals their sum, the component of R in any direction equals the sum of the components of P and Q in that direction. Hence R satisfies the first condition of § 88.

Let an axis perpendicular to the plane of P and Q cut that plane in O, and let OA', OB', OC' be perpendiculars from O on P, Q, and R respectively.

$$\begin{aligned} R \cdot OC' &= (P + Q)\,OC' \\ &= P \cdot OC' + Q \cdot OC' \\ &= P(OA' - A'C') + Q(OB' + B'C') \\ &= P \cdot OA' + Q \cdot OB' + Q \cdot B'C' - P \cdot A'C'. \end{aligned}$$

If θ be the angle between AB and $A'B'$,

$$Q \cdot B'C' - P \cdot A'C' = (Q \cdot BC - P \cdot AC) \cos\theta = 0.$$
$$\therefore\ R \cdot OC' = P \cdot OA' + Q \cdot OB'.$$

Hence the moment of R about this axis equals the sum of the moments of P and Q. It is readily seen that the same is true for *any* axis perpendicular to the first, that is, for any axis in a plane parallel to the plane of P, Q, and R. Hence R satisfies the second condition of § 88.

Hence R is the resultant of P and Q.

The point of application of the resultant, C, is sometimes called the "centre" of the parallel force. Its position is evidently independent of the actual direction of the parallel forces and would not be changed if they were turned in some other direction, while still remaining parallel.

(2) Let the forces P and Q be in opposite directions and let P be $> Q$. Consider a force $R = P - Q$ parallel

to P and Q and applied at a point C in BA produced beyond A such that $P \cdot AC = Q \cdot BC$. R will evidently satisfy the first condition required of the resultant of P and Q.

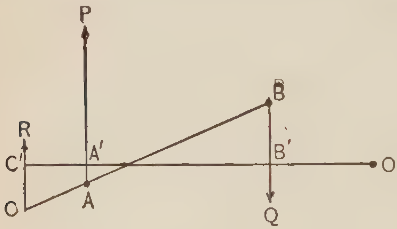

FIG. 49.

Let an axis perpendicular to the plane of P and Q cut that plane in O, and let OA', OB', OC' be perpendiculars to P, Q, and R respectively. Then

$$R \cdot OC' = (P - Q)\,OC'$$

$$= P(OA' + A'C') - Q(OB' + B'C')$$

$$= P \cdot OA' - Q \cdot OB' + P \cdot A'C' - Q \cdot B'C'.$$

If θ be the angle between AC and $A'C'$,

$$P \cdot A'C' - Q \cdot B'C' = (P \cdot AC - Q \cdot BC) \cdot \cos\theta = 0.$$

$$\therefore\ R \cdot OC' = P \cdot OA' - Q \cdot OB'.$$

Hence the moment of R about the axis equals the sum of the moments of P and Q and the same is evidently true for any axis parallel to the plane of P and Q.

Hence R is the resultant of P and Q.

Since
$$P \cdot AC = Q \cdot BC$$
$$= Q(BA + AC)$$
or
$$AC = \frac{Q \cdot BA}{P - Q},$$

it follows that the less the difference between P and Q the farther C is from A or B.

(3) Let P and Q be equal and opposite.

If in the preceding we suppose $P = Q$, then $R = 0$ and $AC = \infty$ or the resultant in a zero force at an infinite distance. This, however, is not a force having any real existence; it is merely a mathematical fiction. No single force can be found that is equivalent to, *i.e.* the resultant of, a pair of equal and opposite parallel forces. Such a pair of forces is called a *couple*.

Parallel forces in the same direction are sometimes called *like* forces; those in opposite directions, *unlike* forces.

90. Resultant of a Number of Parallel Forces. — Let P_1, P_2, $\cdots$ be parallel forces. If they are all in the same direction, P_1 and P_2 are equivalent to a single force R_1 that may be found as in § 89, R_1 and P_3 are equivalent to a single force R_2, and so on. The final result will be a single force R which is therefore the resultant. Evidently $R = \Sigma P$.

If the forces are not all in one direction, those in one direction are equivalent to a single force R_1 and those in the opposite direction to a single force R_2. If R_1 and R_2 are unequal, their resultant is *a single force* $R = \Sigma P$. If R_1 and R_2 are equal, that is if $\Sigma P = 0$, their resultant is a *couple* (or else zero).

From the way in which R is found above it is evident that its point of action is independent of the actual direction of the parallel forces.

A *second method* is to employ the principle that the moment of the resultant about any axis must equal the sum of the moments of the components (§ 88). Let the

distances of the forces from an axis perpendicular to them be p_1, p_2, $\cdots$ respectively. If ΣP is not zero, the resultant is a single force equal to ΣP and its distance, r, from the axis is given by $Rr = \Sigma Pp$. If the same method be applied to a second axis, perpendicular to the forces and to the first axis, it will give the distance of the resultant from the axis and these two distances will give the line of action of R. If all the forces are in one plane, the resultant is in that plane and it will be sufficient to consider moments about a single axis perpendicular to that plane.

A *third method* gives the point of action, C, of R if the points of action of the forces are known. Let the distance of C from any plane be $\bar{x}$, and let the distances of the points of action of the forces from that plane be x_1, x_2, $\cdots$ respectively. Since the position of C is independent of the actual direction of the parallel forces, we may suppose them all turned parallel to the plane referred to. Now take moments about any axis, A, lying in the plane and perpendicular to the new direction of the forces. The distance of the new line of action of P_1 from that axis is x_1, that of P_2 is x_2, and so on. The distance of R from the axis is $\bar{x}$. Hence, $R\bar{x} = \Sigma Px$, and this gives the distance of C from the plane. The same is true for any plane. Hence, taking three planes at right angles, and denoting distances from it by x, y, and z respectively, we have for the position of C

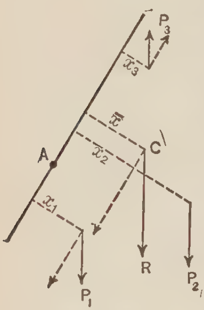

Fig. 50.

$$\bar{x} = \frac{\Sigma Px}{R}, \quad \bar{y} = \frac{\Sigma Py}{R}, \quad \bar{z} = \frac{\Sigma Pz}{R}.$$

If $\Sigma P = 0$, the second and third methods fail; but in that case we may first omit one of the forces and find the resultant of the others. This with the omitted force will form a couple, which is, therefore, the resultant of all the forces.

Exercise XX. Composition of Parallel Forces

(1) *Forces in the same direction.* — A very light framework, consisting of two wooden rods crossed, is supported in front of a cross-section board by an axis that passes through the intersection of the rods. Masses are suspended from the ends of the rods by means of long threads so that they hang below the board. Each thread is attached to the rod by means of a thumb-tack, and hangs over the end of the rod, so that the point of application of the force is sharply defined by an edge of the end section. (There is a small projection on each end of the rear face of the front rod, and the thread is attached to it, so that it may hang clear of the other rod and all the strings may be in the same plane.)

One of the masses is a scale pan carrying weights. The weights are adjusted until the framework hangs in equilibrium, with all the threads at considerable distances from the axis of support. Instead of the scale pan a calibrated spring may be used. The resultant is then found by the first method of § 90, the forces being taken in order around the quadrilateral formed by their points of action. The second and third methods are then applied to find the position and point application of the resultant. All the results should be represented in diagrams.

(2) *Forces in opposite directions.* — Forces acting vertically upward are produced by calibrated springs attached to pegs at the top of the board. Care should be taken that the board is properly levelled and that the lines of action of the springs are truly vertical. A diagram is drawn as before and the resultant found by all three methods.

DISCUSSION

(*a*) How would the results have differed if the forces had been at some inclination to the vertical, the same for all?

(*b*) What should be the relative position of the point of action of the resultant and the supporting knife-edge?

(*c*) Compare the methods of finding the resultant of parallel forces in the same direction and the methods of finding the centre of mass of a number of particles.

(*d*) If parallel forces proportional to their masses be applied to a number of particles, where will the point of action of their resultant fall?

(*e*) How would the framework begin to move if one thread broke?

(*f*) Suppose in (2) the supporting knife-edge were not used. What would be the initial motion if one thread broke?

91. Couples. — The moment of a couple about an axis perpendicular to the couple is constant, that is, independent of the position of the axis. For let O be the projection of any axis. Then if O be between P and Q, the moment of the couple about O is $P \cdot OA + Q \cdot OB = P \cdot AB$. If O be not between P and Q, $P \cdot OA - Q \cdot OB = P \cdot AB$. Hence the moment of the couple is the product of either force by the distance between the forces and is the same about all axes at right angles to the plane of the couple.

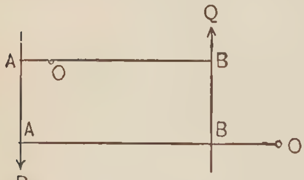

FIG. 51.

If the resultant of all the forces acting on a body is a couple, the motion of the centre of mass will not change (§ 85), but the couple will produce an angular acceleration about the centre of mass proportional to the moment of the couple. Hence it follows that all couples in a plane are equivalent if they have equal moments.

Any length perpendicular to a couple and proportional to the moment of the couple may be used to represent the couple and is called the *axis* of the couple.

It is shown in more advanced works that the resultant of any number of forces applied to a body is a single force and a couple the axis of which is parallel to the line of action of the force. In particular cases either the force or the couple may be zero. (Macgregor's "Kinematics and Dynamics," §§ 479–482.)

92. Centre of Gravity. — If to the particles of a body parallel forces, proportional to the masses of the particles and all in the same direction, be applied, the point of action of the resultant will be at the centre of mass. For if P be the force on a particle m, $P = km$ where k is some constant; and if R be the resultant force and M the whole mass, $R = k \cdot M$. To find the point of action of the resultant we substitute these values in the formulæ of § 90. On doing so the k cancels out and we get formulæ identical with those defining the centre of mass in § 80.

The weights of the particles of a body are forces proportional to the masses of the particles and they are practically parallel, provided the body be of moderate size. The point of action of the resultant is called the *centre of gravity* of the body. It follows from the above that the *centre of gravity of a body coincides with its centre of mass.*

It should, however, be noted that, in general, only a body of moderate dimensions can be said to have a centre of gravity, although bodies of certain particular forms have centres of gravity, no matter what their magnitudes. Every body has a centre of mass.

93. Equilibrium of a Body. — A body is in equilibrium when its motion is constant, that is when the linear velocity of its centre of mass is constant and its angular velocity about any axis is constant. If no forces acted on a body, it would be in equilibrium; but a body may also be in equilibrium when forces act on it. They must, however, satisfy certain relations called the *conditions of equilibrium.*

Given that a body is in equilibrium we may conclude that

(1) *The sum of the components, in any direction, of the forces acting on the body is zero*, since the linear acceleration of the centre of mass is zero.

(2) *The sum of the moments, about any axis, of the forces acting on the body is zero*, since its angular acceleration about any axis is zero.

These conditions may be briefly stated thus:

$$\Sigma F = 0 \text{ in any direction.}$$

$$\Sigma Fp = 0 \text{ about any axis.}$$

Conversely, if both these conditions are satisfied, the body is evidently in equilibrium.

94. Experimental Method of finding the Centre of Gravity. — A body suspended by a cord is acted on by two forces, gravity, acting vertically at the centre of the body, and the tension of the cord. For equilibrium these two forces must be equal and opposite and in the same line. Hence the centre of gravity lies in the line of the cord produced. By suspending a disk by a cord attached in succession to two different points in the margin of the disk and finding the point of intersection of the lines in the disk that coincide in succession with the line of the cord, the centre of gravity of the disk may be located.

Exercise XXI. Equilibrium of a Body

A disk of wood is suspended in a vertical plane in front of a vertical cross-section board by means of a knife-edge driven into the board

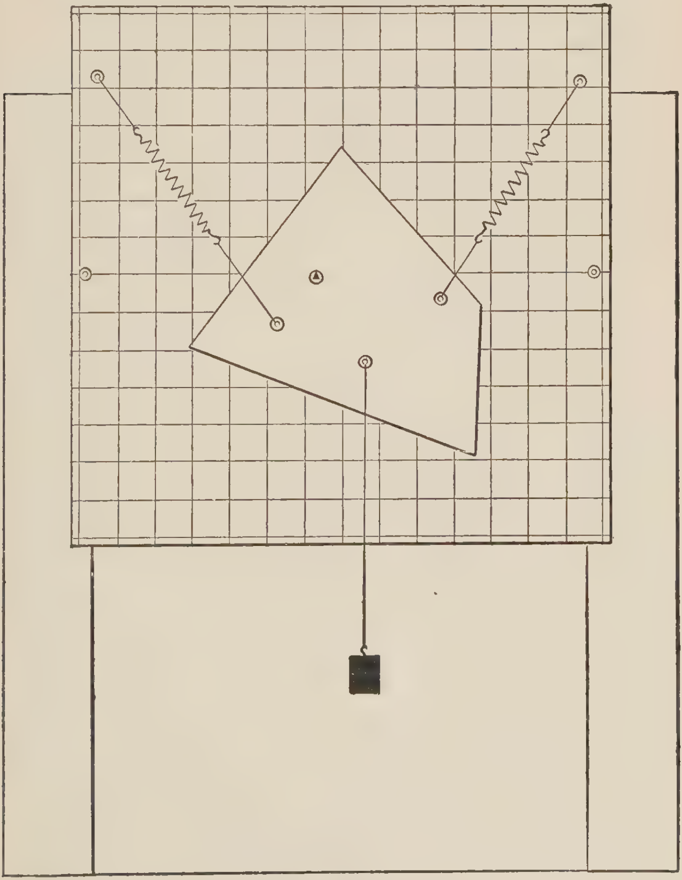

Fig. 52.

and passing loosely through a hole in the disk. Forces are applied to the disk by springs attached as in Exercises XI and XII and by cords

hanging vertically from the disk and carrying weights. The springs should be calibrated before they are used. From previous experience it will be obvious that two points on the calibration curve will suffice.

Since the force of gravity on the disk must be taken into account, the position of the centre of gravity of the disk should be determined by the method of § 94 before the disk is mounted in front of the cross-section board. In setting the apparatus up see that the lines of all the forces are at considerable distances from the axis.

The distance of each force from the axis may be obtained by stretching a long thread very accurately along the line of action of the force. The inclination of the thread to the (positive) horizontal should be measured by a protractor. The length of the springs may be found by the beam compass or mirror scale. From these data calculate the sum of the moments of the various forces (including gravity) about the axis of rotation. The accuracy of the result should be estimated by the *percentage* difference of positive and negative moments.

The magnitude R and direction θ of the reaction of the axis on the disk can be found by equating to zero the sum of the components of all the forces (including the unknown reaction of the axis) first in the horizontal direction and then in the vertical direction,

$$R \cos \theta + \Sigma F \cos \alpha = 0$$
$$R \sin \theta + \Sigma F \sin \alpha = 0.$$

The reaction of the axis may also be found experimentally by removing the axis and replacing it by a peg occupying the same hole in the disk and then attaching to the peg, by a small ring, a cord that passes over a pulley and carries a scale pan. Weights are placed in the scale pan and the position of the pulley adjusted until the disk comes to rest in its former position.

DISCUSSION

(*a*) Graphical method of finding the reaction of the axis.

(*b*) Motion of the centre of gravity of the disk if the axis were suddenly removed.

(*c*) Angular motion of the disk if the axis were suddenly removed.

(*d*) Motion of the disk if the vertical cord were to break.

(*e*) How would the position of the disk alter if board and disk were turned into a horizontal plane, the forces applied remaining unchanged?

95. Special Cases of Equilibrium. — The following special cases of equilibrium are important:

(1) *Two forces.* For equilibrium they must evidently be *equal and opposite and act in the same line.*

(2) *Three parallel forces.* Any one must be equal and opposite to the resultant of the other two. Hence all three must be in the same plane.

(3) *Three non-parallel forces in the same plane.* The moments of any two about the point in which their lines intersect are zero. Hence the moment of the third about that point is zero, that is, its line of action also passes through that point. Hence *all three act through a single point*, and any one is equal and opposite to the resultant of the other two.

(4) *Three forces cannot in any case produce equilibrium unless they act in the same plane.* For consider moments about *any* line that intersects the lines of two of the forces. The moments of these two about any such line are zero. Hence the moment of the third about it is also zero. That is, the third must either be parallel to *every* such line (which is impossible) or it must intersect any such line, and the latter can evidently only be true if the lines of action of all the forces are in the same plane.

Exercise XXII. Equilibrium of a Body

(1) From a uniform iron rod of mass m and length l, a mass M is suspended at a distance h from one end of the rod, and the rod is supported by a horizontal force, F_1, applied to the lower end and a force, F_2, applied to the upper end at an angle of 45° with the horizontal. Find F_1 and F_2, and the inclination of the rod to the horizontal.

This problem is to be solved theoretically, and the result tested experimentally. For the experimental work the rod is suspended in front of

the cross-section board by two springs which apply the forces F_1 and F_2. The springs are attached to small hooks in the end of the rod, the other ends of the springs being borne by pegs inserted in the cross-section board. The cord that sustains M may be fastened to the rod by a slip-noose that binds on the rod, or it may pass through a hole in the rod. The tensions of the springs are deduced from their lengths and calibration curves.

(2) The same rod with its attached weight is held at 30° to the horizontal by a horizontal force at the lower end, and an inclined force at the upper end. Find the magnitudes of the forces and the inclination of the second to the horizontal.

This problem should also be solved both theoretically and experimentally.

DISCUSSION

(*a*) Where would a vertical line that passes through the intersection of the lines of action of the springs intersect the rod? (This might form part of the exercise and be tested experimentally.)

(*b*) Solve the problems (1) and (2) by a method indicated by the answer to (*a*).

(*c*) A uniform beam rests on a smooth horizontal rail and one end of it presses against a smooth vertical wall. In what position will it be in equilibrium?

(*d*) A rod hangs from a hinge on a vertical wall and rests on a smooth floor. Calculate the pressure on the floor and the force on the hinge if the mass of the rod be 1 kg.

(*e*) A uniform ladder 30 ft. long rests with the upper end against a smooth vertical wall, and the lower end is prevented from slipping by a peg. If the inclination to the horizontal be 30°, find the pressure on the wall and at the peg, the ladder weighing 100 lb.

(*f*) A uniform rod is supported by means of two strings which are attached to a fixed point and to the ends of the rod. Show that the tensions of the strings are proportional to their lengths.

(*g*) Three forces acting at the corners of a triangle each perpendicular to the opposite side keep the triangle in equilibrium. Show that each force is proportional to the side to which it is perpendicular.

CHAPTER VIII

WORK AND ENERGY

96. Work. — The scientific conception of work is drawn from many of the most common experiences. One of the oldest of these is the effort put forth in raising a heavy body to a higher level, *e.g.* drawing water from a well in a bucket or carrying stone or brick up a ladder or stair in building a house. The work done in such cases evidently depends on at least two things, — the continued effort required to sustain the weight of the body and the height to which the body is carried. In some cases the work done seems to depend on other circumstances. For instance, a man can raise a quantity of brick to a certain height with less expenditure of work when he uses a pulley than when he actually carries them up. But in the latter case he carries the weight of his body up also and the work of carrying his body up is added to the work of carrying the brick up. Again, more work is required to draw a body up a rough plane to a certain height than up a smooth plane to the same height; but when the plane is rough the force of friction has to be overcome also and the work done against friction is added to the work done against gravity.

A consideration of such cases will show the reasonableness of measuring the *work done* in moving a body by *the*

product of the force applied to the body and the distance through which it is applied.

97. Work and Direction of Motion. — By the distance through which a force is applied is meant the distance measured *in the direction of the force* applied to do the work. When the force overcome is gravity the distance must be measured vertically. A very heavy body can under suitable circumstances, *e.g.* when attached to a crane, be moved horizontally with only a very slight expenditure of work; and the slight amount expended is due to the fact that some friction has to be overcome. If there is no vertical motion, the work done against gravity is zero whatever work may be done against other forces.

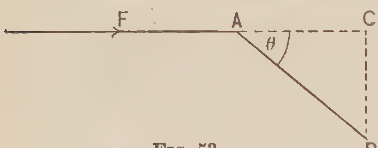

FIG. 53.

Hence if a force F be exerted a distance AB not in the direction of the force, the work done is $F \cdot AB \cos\theta$, θ being the angle between the positive direction of F and the positive direction of AB. But

$$F \cdot AB \cos\theta = F \cos\theta \cdot AB$$

and $F \cos\theta$ is the component of F in the direction of AB. Hence the work done by a force may also be measured by the product of the displacement and the component of the force *in the direction of the displacement.*

If $\theta = 90°$, *i.e.* if the displacement is altogether at right angles to the force, the force does no work. Hence the force towards the centre in circular motion does no work.

98. Units and Dimensions of Work. — The *unit of work* is the work done by unit force when its point of application moves unit distance in the direction of the force. Hence in the absolute C.G.S. system the unit of work is the work done by a dyne when it acts through a centimetre. This unit is called the *erg;* 10,000,000 ergs is called a *joule.* In the F.P.S. gravitational system the unit of work is the work done by a force equal to the weight of a pound when it acts through a foot and is called a *foot-pound.*

Since $W = F \cdot s$, the dimensions of work are

$$(W) = (F)\,(L).$$

Hence in the absolute system

$$(W) = (MLT^{-2})\,(L) = (ML^2T^{-2}).$$

99. Rate of doing Work, or Activity. — The work done by a force depends only on the magnitude of the force and the extent of the displacement, and is *independent of the time* required for the motion. The amount of work an agent does in a certain time, or the *rate of doing work*, is a different thing and is a matter of great importance. (The wealth a man has is the same whether it took him a year or twenty years to accumulate it. The rate at which he can gather wealth is a different thing.)

Rate of doing work is called *activity.* The unit of activity in the absolute C. G. S. system is the activity of an agent that does an erg per second; 10,000,000 ergs per second, or a joule per second, is called a *watt;* 1000 watts is called a *kilowatt.* The F. P. S. gravitational unit of activity is 550 foot-pounds per second, and

is called a *horse-power*. One horse-power = 746 watts very nearly.

100. Diagram of Work. — The work done by a constant force is readily calculated from the force and the displacement. Work done by a variable force, *e.g.* the force exerted by the piston of a steam engine, can be conveniently represented by the following graphical method:

On a straight line, which we may suppose horizontal, lengths are laid off to represent the *displacements in the direction of the force* in intervals so short that the force may be regarded as constant throughout each interval. A vertical line or ordinate is drawn from the middle of each displacement to represent the *magnitude* of the force at the middle of the interval. A smooth curve is then drawn through the upper end of the ordinates. The area between the curve, the horizontal line, and any two ordinates represents the work done in the intervening time. For the work done in one of the short displacements, such as d in the figure, is the product of the force by the displacement, and is therefore represented by the area of the narrow trapezium that stands on d.

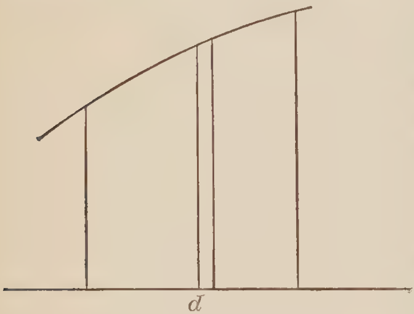

Fig. 54.

101. Energy. — A body capable of doing work is said to possess *energy*, or energy is defined as *capacity for*

doing work. A body can, by descending, draw up another body attached to it by a cord that passes over a pulley. A compressed or extended spring can raise a weight. Water at an elevation can do work in descending to a lower level. These are examples of a body or a system of bodies possessing energy because of some peculiarity in its form or position, or, briefly, because of its *configuration.*

The block of a pile-driver can, when in motion, drive a pile in opposition to the forces of cohesion and friction. A fly wheel in rotation can for a time keep a pump going and raise water. Wind can propel a ship against the resistance of the friction and inertia of the water. These are examples of a body possessing energy because of its *mass and speed.*

Thus we have two chief forms of energy — energy of motion, also called *kinetic energy*, and energy of configuration, also called *potential energy* (potential energy must not be regarded as merely possible kinetic energy, for potential energy can do work without being first transformed into kinetic energy).

102. Kinetic Energy. — The energy a body possesses because of its mass and speed should be capable of being expressed in terms of these quantities. To discover how it should be expressed, let us find how much work a body of mass m and speed v can do in coming to rest. Suppose that the force it exerts on another body is variable, and that while acting through a short distance s_1 it exerts a force F_1. Then the work it does is $F_1 s_1$. By Newton's Third Law, the force opposing its motion is

$-F_1$, and therefore its acceleration is $a = -\frac{F_1}{m}$. If its speed at the end of s_1 be v_1, by § 24

$$v_1^2 = v^2 + 2as_1$$
$$= v^2 - 2s_1\frac{F_1}{m},$$

or
$$F_1s_1 = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_1^2.$$

For successive small displacements s_1, s_2, s_3, $\cdots$, s_n, we have

$$F_2s_2 = \tfrac{1}{2}mv_1^2 - \tfrac{1}{2}mv_2^2,$$
$$F_3s_3 = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_3^2,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$F_ns_n = \tfrac{1}{2}mv_{n-1}^2 - \tfrac{1}{2}mu^2,$$

where u is its final speed.

Adding these equations, we get

$$\Sigma Fs = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2.$$

The left side is the total work done, and if the final speed of the body, u, is zero, its kinetic energy is exhausted, and the whole work it has done is $\frac{1}{2}mv^2$. Hence this was its initial kinetic energy.

In a similar way it can be shown that when a force is opposed only by the inertia of a body, the work done equals the increase of $\frac{1}{2}mv^2$ in the body.

The units and dimensions of kinetic energy are the same as those of work (§ 98).

103. Potential Energy. — Quantities of potential energy are measured by the work they can do. Thus, when the distance of a body from the earth is increased, its potential energy (or more correctly the potential energy of the mass and the earth) is increased by the amount of work

it can do in returning to its first position. The increase of potential energy of a stretched spring is the work it can do in contracting.

We can also express the potential energy of the system consisting of the earth and a body in terms of their masses and distance apart. The increase of potential energy when a body of mass m grams is raised a distance h centimetres (h being small compared with the radius of the earth) is mgh in ergs, for this is the work it can do in returning. The potential energy of a stretched spring can be expressed in terms of its length and elastic constants. But there is no one general way of expressing potential energy as there is in the case of kinetic energy. For each body or system we must find by experiment how much work it can do in changing from one configuration to another and then measure the charge of potential energy by the work so done. If the initial potential energy be V_1 and the final potential energy be V_2, the decrease is $V_1 - V_2$. If the force exerted by the body or system during successive small displacements s_1, s_2, $\cdots$ be F_1, F_2, $\cdots$ respectively, the total work done is ΣFs. Hence when a body or system does work at the expense of its potential energy

$$\Sigma Fs = V_1 - V_2.$$

The units and dimensions of potential energy are the same as those of work (§ 98).

104. Equivalence of Kinetic and Potential Energy. — In the performance of work energy is expended and the energy so expended is by definition equal to the work performed. But when work is done on a system, the energy of the

system is increased. Confining ourselves for the present to cases in which no work is done against friction, or, in other words, supposing the work is wholly performed in producing kinetic or potential energy, then *the energy produced is equal to the work performed, that is, equal to the energy expended.*

The statements in preceding sections will suggest to the reader various ways in which energy can be *transferred* from one body or system to another, the former doing work on the latter. Moreover the energy in a single system may be *transformed* from one form of energy to the other form. Thus when a body is allowed to fall towards the earth, the potential energy of the body and the earth is decreased, but their kinetic energy is (if we neglect friction) increased to an equal amount. In this case the only force affecting the system is an *internal force* between the different parts of the system and *the work done by an internal force results in a transformation of the energy of the system without any change in its amount.*

In making these statements we have supposed that friction may be neglected. As a matter of fact, friction cannot in any case (except possibly in the case of the motion of the heavenly bodies) be wholly neglected. We shall consider later the case of work done against friction.

Exercise XXIII. Energy and Work

Apparatus. — A pendulum with a heavy block of iron as a bob is drawn aside and held in any desired position by a cord arranged as illustrated in Fig. 55. When the cord is released the pendulum falls and impinges on a horizontal rod which is connected to the framework by two horizontal springs. The springs are stretched by the impact and

the pendulum is brought momentarily to rest when it has just reached the vertical. The maximum stretch of the springs is recorded by a

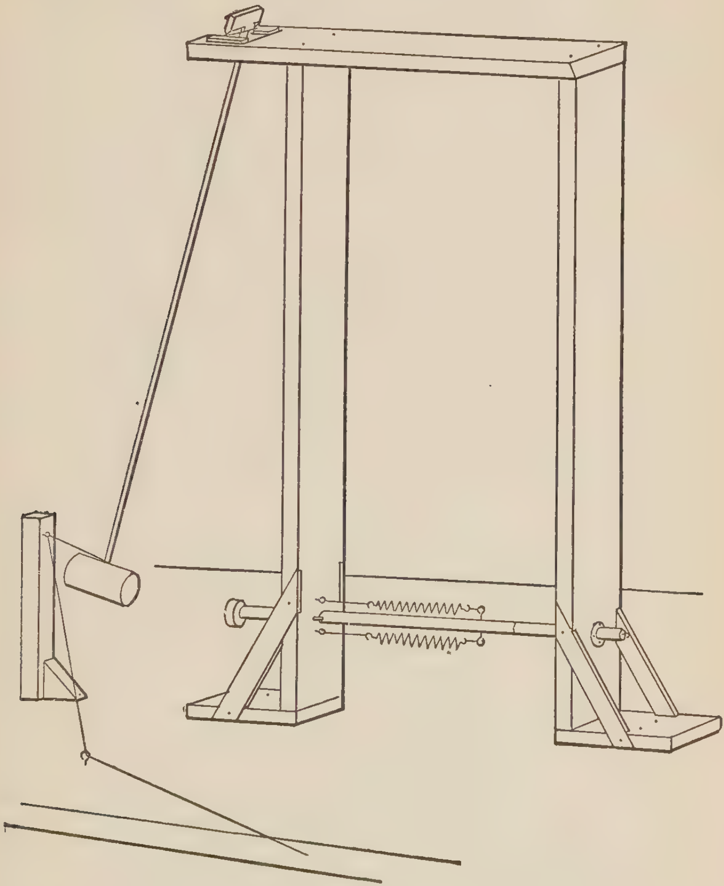

FIG. 55.

loop of thread that is pushed along the rod by a collar in which the rod slides. The rod should be as light as possible; in fact, it is better to use a tube of aluminium instead of a rod. The knife-edge

on which the pendulum swings is carried by two plates, one of which is adjustable by two screws. The necessary readings will be facilitated if a scale is lightly etched on the rod, but this is not indispensable.

Adjustments. — The screws that support the knife-edge are adjusted until the pendulum swings freely without any side motion. The light chains that attach the springs to the framework are adjusted on hooks in the framework until the springs are under slight tension and do not vibrate sidewise when the rod is pushed rapidly by the hand. The proper height from which to release the pendulum so that it just comes to rest in the vertical must be ascertained by several trials. To fix the vertical position of the pendulum, the rod is held by the hand with the springs stretched so that its end just presses against the pendulum when vertical; the thread is then pushed up against the collar and its position noted (or marked in pencil on the rod if there is no scale on the rod). This is the position to which the thread must be moved by the impact if the pendulum comes to rest in the vertical. Before readings are made the rod should be lubricated with vaseline or machine oil.

Measurements. — (1) The length of the pendulum from the knife-edge to the centre of the bob. (2) The length of the cord of the arc through which the centre of gravity of the pendulum falls. (3) The initial length of the springs and the extreme length to which they are stretched; the latter is obtained from the movement of the thread. (4) Calibration of the springs. For this purpose the springs may be removed and calibrated in several steps through the range covered by the experiment. The calibration may also be performed without removing the springs, by attaching to the rod a cord that passes over a pulley and carries a pan and weights.

From the calibration of the springs a diagram of work done by the pendulum in stretching the springs is constructed on cross-section paper and the work calculated. (The scale of the diagram must be taken into consideration as in Exercise IV.) This should nearly equal the loss of potential energy of the pendulum as calculated from its mass and vertical descent, but there will be some difference caused by friction and impact (§ 113).

DISCUSSION

(*a*) Sources of error.

(*b*) Changes that take place in the energy of the pendulum and of the springs.

(*c*) Energy of rotation of the bob of the pendulum.

(*d*) The total energy of the system.

(*e*) Are the results affected in any way by the mass of the rod?

(*f*) Why must the initial tension of the springs not be large?

(*g*) Explain the failure of the pendulum to rise after rebound to its original position.

105. Stable, Unstable, and Neutral Equilibrium. — A body or structure is in equilibrium when the resultant force on it is zero, that is, when it is either at rest or moving uniformly. A body or structure at rest is in *stable equilibrium* when on being displaced it returns (*e.g.* a pendulum, a sphere in a bowl, a chemical balance); it is in *unstable equilibrium* when on being displaced it moves farther away (*e.g.* an egg on one end, a rigid pendulum inverted); it is in *neutral equilibrium* when on being displaced it remains at rest (*e.g.* a sphere on a horizontal plane, a body that can rotate about an axis through its centre of gravity).

If displacement causes an increase of potential energy, it is evident that work has been done against forces opposing the displacement, and these forces will cause the body or structure to return. Hence, *a position of stable equilibrium is a position of minimum potential energy.* The centre of gravity of a pendulum or a balance is raised by a displacement and its potential energy is increased.

If the displacement produces a decrease of potential energy, the forces acting must have aided the displacement, and they will therefore still further increase the

displacement. Hence, *a position of unstable equilibrium is a position of maximum potential energy.* If an egg be supposed balanced on one end, its centre of gravity will be lowered by a displacement, and therefore the potential energy will be decreased.

Finally, if when a body or structure is displaced there is *no change of potential energy*, then the forces acting on the body neither oppose nor assist the motion, and hence the equilibrium will be neutral.

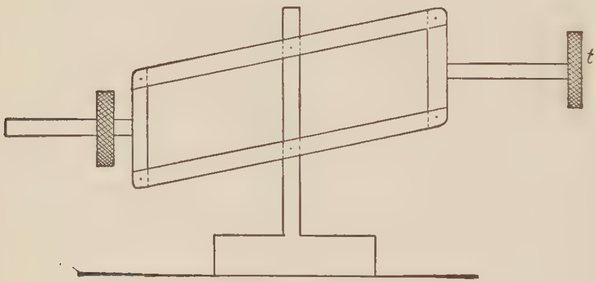

FIG. 56.

The energy criterion of equilibrium may be illustrated by the apparatus represented in the diagram. A light jointed parallelogram is attached to an upright post by horizontal axes passing through the centres of two opposite sides. Equal heavy weights are movable along rods fastened at right angles to the vertical sides. The apparatus is in neutral equilibrium no matter where the weights may be on the rods, for no work is done, and there is no change of potential energy during a displacement.

106. Kinetic Energy of a Rotating Body. — When a body rotates about a fixed axis with angular velocity ω, a par-

ticle, m, at a distance, r, from the axis has a linear velocity ωr, and therefore its kinetic energy is $\frac{1}{2}\ m\omega^2 r^2$. The kinetic energy of the whole body is therefore $E = \Sigma\ \frac{1}{2}\ m\omega^2 r^2$. But since ω is the same for all the particles,

$$E = \tfrac{1}{2}\ \omega^2 \Sigma m r^2 = \tfrac{1}{2}\ I\omega^2,$$

I being the moment of inertia of the body about the axis of rotation. The similarity of this formula to the formula, $\frac{1}{2}\ mv^2$, for the kinetic energy of translation of a body should be noted.

Exercise XXIV. Kinetic and Potential Energy

In Exercise XVII a mass descended losing potential energy, and in so doing it set into rotation a disk which acquired kinetic energy. The descending mass also gained some kinetic energy. The circumstances were such that the resistance of friction was very small. Hence, the gain of kinetic energy should be (at least very nearly) equal to the loss of potential energy. These quantities can be calculated from the observations then made. It may, however, be well to repeat the measurements of distance and time of descent with all the care possible.

To calculate the kinetic energy the final angular velocity must be known. This can be deduced from the time of descent and the distance of descent. For these give the mean velocity of descent, and twice this is the final velocity. From the final linear velocity of descent and the radius of the axis, the final angular velocity of the disk is deduced. Thus we have all the data necessary to test the equality of loss and gain of energy.

(The apparatus of Exercise XVIII may also be used for this experiment, the final velocity of rotation being obtained directly by means of the recording disk. On account of the greater air friction the results will probably not be found as satisfactory as those obtained by the method described above.)

DISCUSSION

(*a*) Sources of error.

(*b*) Is the friction necessarily assumed to be zero in the method of finding the final velocity?

(*c*) Find an expression for the final kinetic energy that does not contain the final linear velocity; also one that does not contain the final angular velocity.

(*d*) What would the final angular velocity be if the thread were wrapped around the disk?

(*e*) Given the coefficient of friction between the cylinders and the disk, at what angular speed would the former slide if not restrained by pegs?

(*f*) If the friction between the cylinders and the disk prevented slipping, at what angular speed would the cylinders be overturned?

(*g*) What form of cylinders would render slipping and overturning equally probable?

(*h*) Method of measuring the coefficient of static friction suggested by this exercise.

(*i*) If the thread remained attached to the axis and the weight just touched the floor, how much would the angular velocity decrease when the thread began to rewind?

(*j*) If the thread remained attached to the axis and the weight did not touch the floor, how much would the angular velocity decrease when the cord began to rewind?

107. Simultaneous Translation and Rotation. — We have already seen that the motion of a body may be considered as consisting of velocity of translation of the centre of mass, and velocity of rotation about the centre of mass, and that these two motions may be regarded as independent, and may be calculated separately (§ 86). Similarly the whole kinetic energy of a body may be considered as consisting of two parts corresponding to these two motions. If M is the whole mass of the body, and $\bar{V}$

the velocity of the centre of mass, the kinetic energy of translation is $\frac{1}{2}M\overline{V}^2$. If the body has an angular velocity, ω, about an axis through the centre of mass, and if I is the moment of inertia about that axis, the kinetic energy of rotation is $\frac{1}{2}I\omega^2$. The total energy is the sum of these two. Thus the kinetic energy of a locomotive wheel can be calculated when the velocity of its centre, and its angular velocity about its centre, are known.

The same principle applies to the motion of any group of particles, but, if the particles are not connected rigidly together, they have no angular velocity in common, and their kinetic energy, relatively to the centre of mass, must be found in a different way. Suppose that any particle, m, has, relatively to the centre of mass of all the particles, a velocity V; its kinetic energy of motion, relatively to the centre of mass, is $\frac{1}{2}mV^2$. Summing up for all the particles, we get the expression $\Sigma\frac{1}{2}mV^2$ for the kinetic energy of all, relatively to their centre of mass. This, together with $\frac{1}{2}M\overline{V}^2$, makes up the whole kinetic energy of the group of particles.

Since internal forces do not affect the motion of the centre of mass of a body or group of particles (§ 85), they cannot change the part of the kinetic energy that depends on the motion of the centre of mass.

The propriety of dividing kinetic energy into these two parts needs, in reality, somewhat more proof than has been given. Suppose the velocity $\overline{V}$ of the centre of mass to be resolved into rectangular components $\overline{u}$, $\overline{v}$, $\overline{w}$. Also let V, the velocity of any particle, m, relatively to the centre of mass, be resolved into components, u, v, w. Then the whole velocity, relatively to the origin, of the particle, m, has components $\overline{u}+u$, $\overline{v}+v$, $\overline{w}+w$, and its square is therefore equal to the

sum of the squares of these components. Hence the whole kinetic energy is

$$E = \Sigma \tfrac{1}{2} m\{(\bar{u} + u)^2 + (\bar{v} + v)^2 + (\bar{w} + w)^2\}$$
$$= \Sigma \tfrac{1}{2} m(\bar{u}^2 + \bar{v}^2 + \bar{w}^2) + \Sigma \tfrac{1}{2} m(u^2 + v^2 + w^2)$$
$$+ \bar{u}\Sigma mu + \bar{v}\Sigma mv + \bar{w}\Sigma mw.$$

The last three terms are all zero since $\Sigma mu = \Sigma mv = \Sigma mw = 0$ (§ 84). The first term is readily seen to be $\frac{1}{2} M\bar{V}^2$, since $\bar{u}$, $\bar{v}$, $\bar{w}$, are the components of $\bar{V}$. The second term is $\Sigma \frac{1}{2} mV^2$, and in the case of a rigid body this is equal to $\frac{1}{2} I\omega^2$ (§ 106).

108. Work done by the Moment of a Force or Couple. — A force, F, acting at a point, P, of a body free to rotate about an axis, A, will produce a displacement of P and will therefore do work. Let f be the component of F in the direction of motion of P, and let r be the distance of P from the axis A.

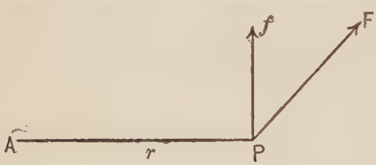

FIG. 57.

If F be of constant magnitude and if its direction with reference to AP be also maintained constant, then f will be of constant magnitude. When the body turns through an angle θ, P will move through a distance $r\theta$, and the work done by F will be $fr\theta$. But since f is the only component of F that has a moment about A, the moment of F about A is fr. Denoting it by C, the work done by C in an angular displacement θ will be $C\theta$. If the force be entirely exerted in producing kinetic energy of rotation about A, then

$$C\theta = \tfrac{1}{2} I\omega_1^2 - \tfrac{1}{2} I\omega^2,$$

ω being the initial angular velocity, and ω_1 the final angular velocity.

If F be not constant or if its direction with reference to AP vary, then f and also C will be variable. In this case the work done will be $\Sigma C\theta$, C being the moment of the force during a small displacement θ.

Exercise XXV. Kinetic and Potential Energy

In Exercise IX no account was taken of the mass of the wheel nor of the force required to overcome the friction of the bearings of the wheel. It is now proposed to take account of these and treat the problem from the point of view of gain and loss of energy. The observations described under "distance and acceleration" in Exercise IX may be repeated; or, if exactly the same apparatus is used, the results obtained in that exercise may be used; but the former is preferable, since doubt would remain as to whether the apparatus was in exactly the same condition in all respects.

Loss of Potential Energy. — The resultant loss of potential energy is readily calculated from the loss of potential energy of the total descending mass and the gain of potential energy of the ascending mass. The items should be stated in ergs or joules.

Kinetic Energy of Masses. — From the acceleration and distance the final velocity of the masses can be calculated and thence the kinetic energy deduced.

Friction. — The friction of the bearings of the wheel varies somewhat with the position of the wheel. It may be measured though not very accurately by finding what additional weight placed on one of the masses will just keep the (equal) masses in steady motion when once started. This should be repeated for several different positions of the cord on the wheel and the mean taken. The work done against friction equals the product of the moment of the force that will overcome the friction and the total angle through which the wheel turns.

Kinetic Energy of Wheel. — The kinetic energy gained by the wheel can be calculated from its moment of inertia and final angular velocity. The final angular velocity of the wheel is readily deduced from the relation between the angular velocity of the wheel, the linear velocity

of a point on its circumference (which is the same as the linear velocity of the cord), and the radius of the groove on the wheel.

The moment of inertia of the wheel may be found by observing the angular acceleration that a known moment of force will give to the wheel. Remove the large masses and the wire and wrap a light cord around the wheel, fastening one end to a spoke. To the free end attach a weight and find the height to which the weight must be raised so that when the wheel is released on a tick of the clock, the weight will just strike the floor after some exact number of seconds. To allow for friction, find as before what small weight attached to the cord will just keep the wheel in steady motion.

109. Conservative and Dissipative Forces. — In stating the equivalence of kinetic and potential energy during transference or transformation, we limited the statement to cases in which energy is expended while exerting forces that are wholly employed in producing kinetic or potential energy. In such cases the total quantity of kinetic and potential energy is unchanged, or, as it is frequently stated, *conserved*. The forces through whose agency such transference and transformation are effected are called *conservative* forces. Examples are the force of gravitation and the force exerted by a compressed spring.

If we examine such conservative forces, we shall find that they have one characteristic in common. They are known when the positions or configurations of the bodies are assigned. The force of gravitation on a body at a certain height is the same whether the body be at rest or moving in any way. The force exerted by a spring depends only on its length at a certain moment and not on whether it is contracting or expanding.

Why the energy should remain constant when the only forces acting are conservative forces can be readily seen.

Consider a body started upward along a smooth plane. While rising it is acted on at each point by a force of a certain magnitude, and so does work and therefore loses kinetic energy to an extent depending only on the force at each point and the displacement. But exactly the same force will act on it at each point when it descends, and therefore while rising it is gaining power of doing work or potential energy equal to the work done or kinetic energy lost in rising. Hence its total energy remains constant.

But now suppose the same body started up a rough plane. Then a second force, friction, acts on it. This opposes its rise and will also oppose its descent. Hence the resultant force on the body will be less at any height during the descent than it was during the rise. Therefore, in rising it does not accumulate power of doing work in the form of potential energy equal to the loss of kinetic energy.

Friction is thus a non-conservative, or, as it is frequently called, a *dissipative* force. It depends on the direction (and often on the magnitude) of the velocity of the body on which it acts. Other dissipative forces more or less analogous to friction will be met with in the special branches of physics.

For an obvious reason conservative forces are sometimes called *positional* forces, and dissipative forces are sometimes called *motional* forces.

110. The Conservation of Energy. — A system of bodies that neither does work on outside bodies nor has work done on it by outside bodies is an *isolated system.* While

no system is completely isolated, many are nearly so, and may for most purposes be regarded as isolated. The whole solar system is perhaps the best example. Except for the atmosphere, the earth and a projectile might be treated as an isolated system. The earth, a projectile, and the atmosphere is a still closer approach to an isolated system. A spring or tuning-fork vibrating in a vacuum would be almost an isolated system.

When an isolated system does work against internal dissipative forces, the amount of work so done represents an equal amount of kinetic or potential energy subtracted from the system. Hence, *if to the sum of the kinetic and potential energies of an isolated system we add the work done against dissipative forces, the whole sum is constant.*

What becomes of the energy spent in doing work against dissipative forces? It was long supposed to be wholly annihilated. Newton was aware of the constancy of the sum of what we now call kinetic and potential energy and work done against dissipative forces, but it was nearly two centuries before it was recognized that the work done against dissipative forces gives rises to a store of energy equal to that expended in doing the work. It was then found that such work produces an amount of *heat*, that, measured in heat units, is in all cases exactly proportional to the energy expended, or, in other words, that *heat is equivalent to energy and is therefore itself a form of energy*. Previous to that time, heat was supposed to be a very light form of matter called *caloric*.

The new view naturally suggested that, if we could follow the motion of the particles of a body that becomes heated, we would find that each particle has a store of

kinetic and potential energy, and that, moreover, in the changes from one form to the other and from particle to particle only conservative forces come into play. When a body slides down a rough plane, the change of some of the energy of motion of the whole body into energy of motion of the separate particles of the body and of the plane may be effected by conservative forces between the particles. From this point of view the distinction between conservative and dissipative forces would disappear.

But the belief that heat is a form of energy is not founded on any view as to the state of the particles of a body that is heated. The belief is founded on the fact that heat and other forms of energy are interchangeable and numerically equivalent. Including heat as a form of energy and also other forms of energy that will be treated in the special branches of physics we may state the law of the *Conservation of Energy* thus: "The total energy of any material system is a quantity which can neither be increased nor diminished by any action *between the parts of the system*, though it may be transformed into any of the forms of which energy is susceptible" (Maxwell).

111. The Dissipation of Energy. — We have seen that from one point of view the distinction between conservative and dissipative forces seems to disappear. But from another point of view there seems a more permanent distinction between them. Work done against conservative forces produces forms of energy that can be confined to definite portions of matter. For example, the potential energy of the earth and a heavy body seems to remain definitely associated with them, and the potential energy

of a distorted spring seems to have no tendency to escape. On the other hand, work done against dissipative forces gives rise to forms of energy that tend to diffuse without limit. The heat produced by work against friction spreads from molecule to molecule of the bodies in contact and from them to other adjacent bodies. This diffusion of energy wherever dissipative forces are in action is called the *dissipation of energy*. It is a process always going on, for dissipative forces are present wherever changes of any kind are taking place in Nature.

The principle of the dissipation of energy must not be regarded as at variance with the law of the conservation of energy. The former refers merely to the *distribution* of energy, the latter to the *constancy of the quantity* of energy.

112. Impact. — When two spheres moving in the line joining their centres impinge, there are two stages to the impact: (1) they compress each other until they come relatively to rest; (2) they then recover partially or wholly and push one another apart. During both stages they repel one another with forces that are by Newton's Third Law equal and opposite. Hence they suffer equal and opposite changes of momentum, or *the total momentum is unchanged by the impact.*

A simple relation between the relative velocities of the spheres before and after impact was discovered experimentally by Newton. Let us suppose that the spheres are of the same material, are *homogeneous*, that is, have the same properties at all parts of their mass, and are *isotropic* or have at any point the same properties in all

directions. Then the *ratio of the velocity of separation after impact to the velocity of approach before impact is constant*, that is, is independent of the sizes of the spheres and their separate velocities, and depends only on the material of which they consist. This law has been shown by others to hold true only within certain limits of velocity. The ratio of the velocity of separation to the velocity of approach is called the "coefficient of restitution" of the material.

If the spheres are not homogeneous, the coefficient of restitution depends on the properties of the spheres at the points of contact. If they are not isotropic (*e.g.* wood), the coefficient depends on the direction of the grain relatively to the line of impact.

Let the masses of the spheres be m_1 and m_2, their respective velocities before impact u_1 and u_2, and after impact v_1 and v_2; then from the constancy of the total momentum we have

$$m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2. \qquad (1)$$

Consider the case in which m_1 is ahead before impact and both m_1 and m_2 are moving in the positive direction. The velocity of approach is $u_2 - u_1$, and of separation, $v_1 - v_2$. Hence, by definition of the coefficient of restitution,

$$e = \frac{(v_1 - v_2)}{(u_2 - u_1)},$$

or

$$v_1 - v_2 = - e(u_1 - u_2). \qquad (2)$$

From (1) and (2), v_1 and v_2 can be calculated. Care must be taken to give the numerical values of u_1 and u_2 their proper signs.

If the impact of two smooth spheres be oblique, that is, if the spheres be not moving before impact in the line of their centres, then, since the pressure between the spheres is altogether in the line joining their centres, only the components of their velocities along that line will be affected, and the above equations will apply to those components only. The components perpendicular to the line of the centres will be unchanged.

113. Dissipation of Energy on Impact. — The forces that come into play during impact are not wholly conservative. If the material of the impinging bodies is plastic, as in the case of putty or lead, the forces tending to produce separation are small, and e is small. The kinetic energy of the system is partly spent in deforming the materials, doing work against cohesion and internal friction. Even if the materials recover wholly from deformation, still, during the deformation and recovery, work is done against internal friction, and heat and sound are produced. Thus energy is dissipated and the kinetic energy after impact is less than that before. The amount so dissipated can be found from the masses and their velocities.

Exercise XXVI. Impact

Apparatus. — A ball of ivory or wood forming the bob of a pendulum impinges on another ball suspended similarly and initially at rest. Each supporting cord is in the form of a V, in order that the motion of the ball may be confined to a vertical plane. The impinging ball should be dropped from some definite position without any jar at starting. To facilitate this, the apparatus is provided with a rod, in the end of which a needle is thrust; over the end of the needle a small loop of thread attached to the ball is passed. When the ball has come to rest, the loop of thread is gently pushed off the needle,

and so the ball is released with very little jar. A block carrying two vertical wires (or knitting needles) and movable along a horizontal

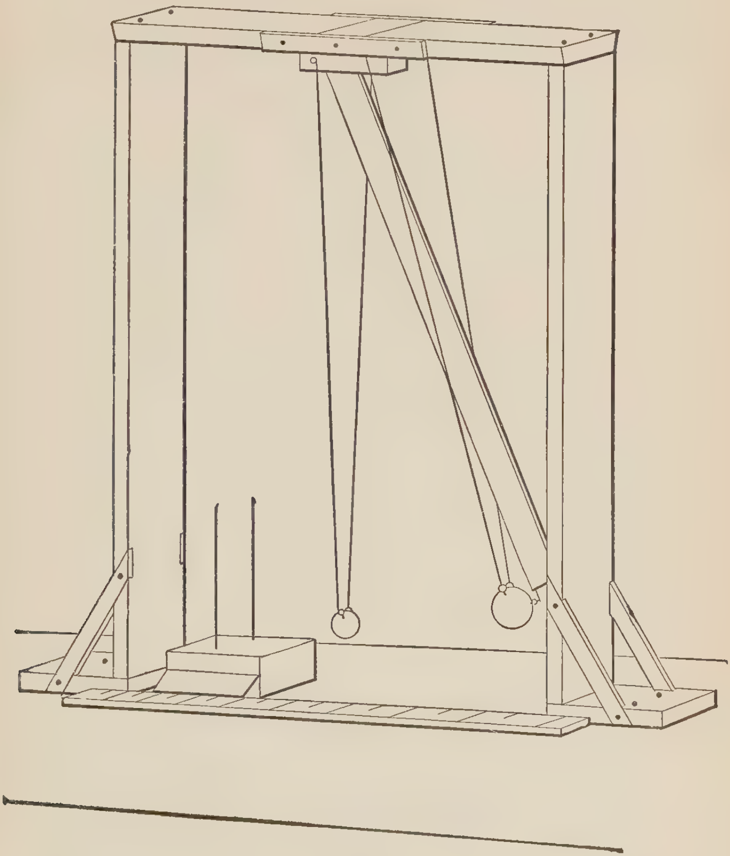

FIG. 58.

metre stick is used to measure the horizontal distances through which each ball moves. The wires should be made vertical by a square.

Adjustments. — The supporting cords are adjusted until the impinging ball is moving, at the moment of impact, in the line joining the

centres of the two balls and the second ball moves off in the same line. The distance between the upper ends of the cords must be made equal to the sum of the radii of the balls so that when the pendulums are at rest the balls just touch without pressing against one another.

Observations. — The velocity of each ball before impact and also its velocity after impact are to be found. The velocity can be deduced if the height from which each ball falls or the height to which it rises is known. The height is too small to be measured directly with any degree of accuracy; but the horizontal distance traversed can be measured with considerable accuracy by means of the movable block and the horizontal scale; and, from the horizontal distance and the radius of the circle of motion, the height can be deduced.

In the case of the impinging ball the measurement of the horizontal distance is made by adjusting the movable block until the ball is tangential to the plane of the upright wires, (1) when the ball is in its elevated position, and (2) when the ball is at rest in its lowest position, and reading the position of the block on the scale in each position. To obtain the height to which each ball rises after impact the movable block is adjusted until the ball as it rises just seems to touch the plane of the wires. Several trials will be needed in each case to accomplish this adjustment.

In this way the velocity of the impinging ball before impact and the velocity of each ball after impact are measured. From these velocities and the masses of the balls (which can be obtained by weighing) the total momentum and the total kinetic energy before impact and the total momentum and total kinetic energy after impact are calculated. The coefficient of restitution should also be calculated.

The experiment should be performed both with balls of equal masses and with balls of unequal masses.

DISCUSSION

(*a*) Meaning and derivation of formulæ.

(*b*) Sources of experimental error.

(*c*) Deduction of the ratio of the masses of two bodies from the changes of velocity when one impinges on the other. Distinction between mass and weight.

(*d*) What becomes of the energy lost on impact?

(*e*) How is the motion of the centre of gravity of the two balls affected by the impact?

(*f*) A 50-gm. bullet is fired into a ballistic pendulum whose mass is 50 kg. If the velocity of the bullet is 300 m. per second, what is the velocity and momentum of the pendulum?

(*g*) Find the loss of kinetic energy in (*f*).

(*h*) A ball falls from a height of 16 ft. and rebounds from a stone slab to a height of 8 ft. Find the coefficient of restitution.

(*i*) A Maxim gun fires 5 bullets per second each of mass 30 g. and having an initial velocity of 500 m. per second. What force is necessary to hold the gun at rest?

(*j*) To what extent is Exercise XXIII a case of impact?

114. Dissipation of Energy of Rotation. — Exercise XVIII, on the constancy of the angular momentum of a system whose moment of inertia changes, affords an illustration of the dissipation of energy of rotation. If the moments of inertia before and after the change be I_1 and I_2 respectively, and the angular velocities ω_1 and ω_2 respectively, then the corresponding values of the kinetic energy are $\frac{1}{2} I_1\omega_1^2$ and $\frac{1}{2} I_2\omega_2^2$. Since $I_1\omega_1 = I_2\omega_2$, the ratio of the kinetic energy after the change to that before the change is $\omega_2 : \omega_1$.

The energy lost is expended in work against the friction between the sliding blocks and the cross-arm and in the production of heat and sound on the impact of the blocks on the stops. To draw the blocks back again to their original positions, thus increasing the angular velocity and restoring the lost kinetic energy, work would have to be done equal to the energy dissipated. To accomplish this a force would have to be applied to the cord equal, at each stage, to the sum of the centrifugal force and friction and

the total work done by the force applied would equal the increase of kinetic energy.

If during the outward movement of the blocks they were compelled to draw up a weight and if the weight were of such magnitude that the blocks just reached the stops without impinging, the total kinetic and potential energy of the system including the weight would be unchanged except for the energy expended against friction. Or the blocks might be compelled to stretch springs attached to the vertical axis, and then the potential energy of the springs would (neglecting friction) be equal to the loss of kinetic energy.

Exercise XXVII. Angular Momentum and Kinetic Energy of Rotation

The change of kinetic energy that accompanied the change of moment of inertia in Exercise XVIII can be calculated from the initial and the final moment of inertia and the initial and the final angular velocity as indicated above. It can also be found by the following experimental method.

Let the kinetic energy before the change of moment of inertia be E_1. Attach the thread (see Fig. 46) to the axis so that it will not be detached when wholly unwrapped and arrange the cord that restrains the blocks so that the moment of inertia shall not change. Then the thread after unwinding will be rewound on the axis and the weight m will rise finally to a height h. This will be less than the height H, from which m originally descended owing to the effects of friction. If W be the work done against friction during the ascent of m, $E_1 = mgh + W$. If the experiment be repeated, the cord having been arranged so that the moment of inertia changes, m will rise to a height, h', much less than h owing to the decrease of kinetic energy that takes place when the moment of inertia changes. Moreover, the work done against friction will be less in this case since the total amount of ro-

tation will be less. But suppose that m at the lowest point of its descent is replaced by a smaller mass m', such that the moment of inertia having changed, m' is finally raised to the height h. Then, the total rotation being the same in the two cases of ascent to the height h, the work done against friction will be the same, namely W. Hence, if the kinetic energy after the change of moment of inertia be E_2, we shall have $E_2 = m'gh + W$. Hence $E_1 - E_2 = mgh - m'gh = (m - m')gh$, $m - m'$ being the amount by which m was supposed decreased. The magnitude of $m - m'$ can be ascertained by the following arrangement.

Let m be replaced by two scale pans, one attached below the other in such a way that the lower one becomes detached when it touches a platform placed below it. (The method of attachment is indicated in Fig. 59.) Let weights be placed in both pans and let the total mass (including the pans) be m. The proper distribution of weights between the two pans so that, the lower having become detached at the lowest point of descent, the upper one will rise again to the height h, can be found after a few trials. A little thought will show how after but one trial the desired distribution can be roughly predicted. A second trial with this predicted value will give a still closer approximation, and so on. Four or five such trials will suffice to accomplish the object sought.

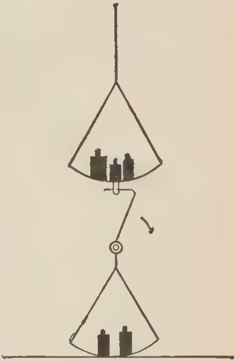

FIG. 59.

To compare this experimental value of $E_1 - E_2$ with the calculated value, the values of ω_1 and ω_2 must be known. These may be taken from the results of Exercise XVIII provided the apparatus and conditions be the same. To test the latter the various heights H, h, h' should be tested and if found the same it will not be necessary to redetermine ω_1 and ω_2. (It may be noted that there is very little difficulty in putting up the apparatus for all practical purposes exactly as it was in Exercise XVIII. The only difference to be feared is in the friction of the bearings and this friction is in reality very small com-

pared with the air-resistance on the rotating parts.) The times of ascent and descent should also be ascertained and noted.

In the above the *loss* of kinetic energy has been found experimentally, but the initial and final values of the kinetic energy can also be found. First for E_1 we have

$$E_1 = mgH - FH \text{ (descent)},$$
$$E_1 = mgh + Fh \text{ (ascent)},$$

Hence $$E_1 = 2\,mg\,\frac{Hh}{H+h}.$$

E_2 can be found in a similar way if the height H' from which m must descend in order that it may reascend to the height h' be found experimentally

$$E_2 = mgH' - F'H' \text{ (descent)},$$
$$E_2 = mgh' + F'h' \text{ (ascent)},$$
$$\therefore\ E_2 = 2\,mg\,\frac{H'h'}{H'+h'}.$$

Other points that might be examined experimentally will be suggested below.

DISCUSSION

(*a*) Sources of error.

(*b*) Suggestions for the improvement of the apparatus.

(*c*) What weight could the blocks raise while sliding out?

(*d*) What becomes of the kinetic energy lost on change of moment of inertia?

(*e*) Is it justifiable to assume F and F' equal?

(*f*) Can F and F' be found directly by experiment?

(*g*) What percentage error would there be in the calculated values of E_1 and E_2 if the pendulum were not exactly a second's pendulum?

(*h*) Calculate ω_1 from m, the time of descent H, and the radius of the axis. Account for the calculated value not agreeing exactly with the experimental value.

(*i*) There is no difficulty in arranging the apparatus so that the sliding blocks are released when only half of the thread has unwound. This being done, what will the velocity be when the whole thread has unwound?

(*j*) Analogy between this exercise and the one on impact.

In Fig. 60 is shown an apparatus for illustrating qualitatively the conservation of angular momentum when moment of inertia and angular velocity change and the associated changes of kinetic energy

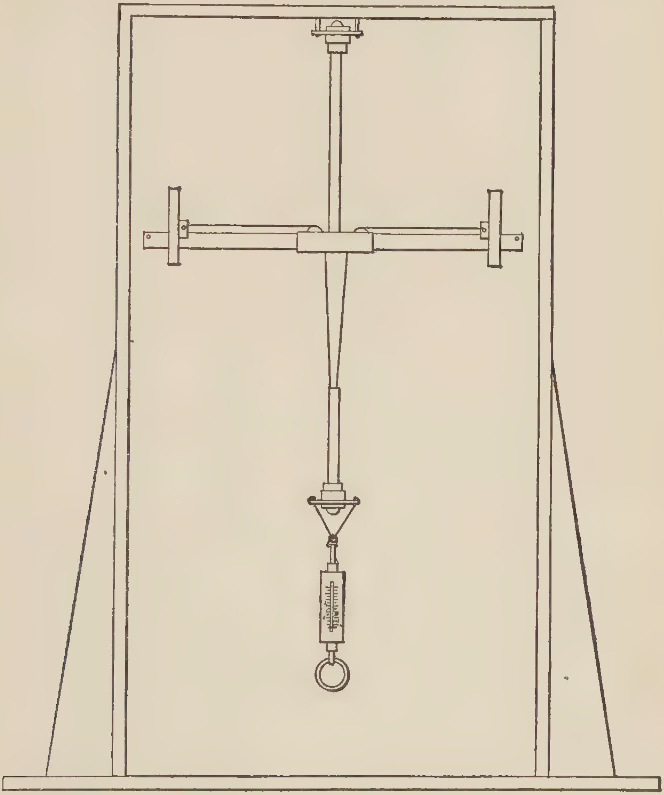

Fig. 60.

and centrifugal force. The vertical brass tube that supports the horizontal rod is carried by a ball-bearing and the cord attached to the sliding weight is connected to another piece of tubing which supports a spring balance by a ball-bearing (both bearings may be

made from a bicycle pedal). If the sliding blocks are massive compared with the horizontal rod, the changes of kinetic energy and centrifugal force may be stated very simply. Let m be the mass of each block and r its distance from the axis of rotation. The total angular momentum $M = 2\,mr^2\omega$; the total kinetic energy $E = mr^2\omega^2$, and the centrifugal force $F = m\omega^2 r$. Since M remains constant when r changes, by eliminating ω we find that $E \propto \frac{1}{r^2}$, and $F \propto \frac{1}{r^3}$. If r be reduced to one-half by downward pressure on the ring of the balance, F will be increased eightfold, as will be fairly well shown by the balance, provided the horizontal rod be well lubricated. A horizontal scale with a movable pointer may be attached to the framework just below the rotating arm in order to indicate the position of the blocks.

The sudden changes of angular velocity and of centrifugal force are very striking.

115. Notes on Some Difficulties. — Kinetic and potential energy are equivalent. One can be changed into the other. But while kinetic energy involves motion, potential energy is energy of bodies which may be relatively at rest. It is somewhat difficult to understand how the energy of something at rest should be equivalent to energy that depends on motion. Some progress is, however, being made in the direction of explaining potential energy as being really kinetic energy of particles too small to be separately visible. Thus the heat energy of a body is believed to be kinetic energy of the particles of the body. The potential energy of a spring may also at basis be kinetic energy of particles separately invisible, but attempts at a definite explanation have hitherto been fruitless.

That the potential energy of an elastic solid may conceivably be energy of "concealed motion" may be illustrated by the apparatus shown in Fig. 60. Suppose the rotating parts covered up so as to be invisible. To "stretch" the apparatus by pulling on the cord,

work is required, and the apparatus will do the same amount of work in "contracting," thus imitating to a certain extent the action of a spring. A person ignorant of the mechanism might have to be content to describe the energy of the apparatus as potential energy, whereas it is really kinetic energy of the rotating masses. This must not be understood as anything more than a crude illustration of the statement that all potential energy may be really kinetic energy of invisible parts.

The kinetic energy of a body is calculated from its mass and velocity. Now the velocity of a body means its velocity relatively to some point taken as origin (§§ 6, 18), usually a point on the surface of the earth. Thus by kinetic energy we can never mean anything but energy of relative motion of bodies. The choice of a point on the surface of the earth seems arbitrary. Would it affect our calculation if we should choose the centre of the earth or the centre of the sun or a star as origin? To answer this we may note that it is in reality only with *changes* of kinetic energy that we are concerned. Consider for example the impact of two bodies (Exercise XXVI). Their kinetic energy before impact may be considered as consisting of two portions (§ 107), E_1, or the energy of their motion relatively to their centre of mass C and E_2, or the energy due to the motion of C relatively to the point taken as origin. The impact does not change the motion of C (§ 85) and hence does not alter E_2. Thus the loss of kinetic energy is the decrease in E_1 which is independent of the choice of origin. Similar considerations apply to other cases.

REFERENCES

Balfour Stewart's "Conservation of Energy."

Maxwell's "Matter and Motion."

Daniell's "Principles of Physics," Chapter IV.

CHAPTER IX

PERIODIC MOTIONS OF RIGID BODIES

116. Angular Simple Harmonic Motion.—Linear S. H. M. is a vibration in a line according to the law $a =$ constant $\times x$. A rigid body free only to rotate may have a closely analogous motion. For instance, the balance-wheel of a watch rotates first in one direction and then in the opposite direction, its excursions (when the motion is steady) being confined to a certain angle.

Angular S. H. M. may be defined as the motion of a body that vibrates through an angle in such a way that the angular acceleration α is always opposite to and proportional to the angular displacement θ, or so that

$$\alpha = -A \cdot \theta, \tag{1}$$

A remaining constant throughout the motion.

The meaning of the constant A can be found by considering the motion of a point P in the body. Let O be the projection of the axis of rotation. Then P performs vibrations in an arc of a circle of radius r. If θ be the angular displacement of the body at any moment, the arc through which P is displaced from its mean position is $x = r\theta$.

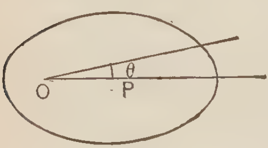

FIG. 61.

If α be the angular acceleration of P at any moment, the linear acceleration of P along the arc is $a = r\alpha$ (§ 30).

Hence the acceleration and displacement of P are connected by the relation

$$\frac{a}{r} = -A \cdot \frac{x}{r},$$

or

$$a = -A \cdot x.$$

Now let us suppose that a point, Q, vibrates in a straight line, and has at each moment the same displacement from a point in the straight line as P has along the arc of vibration, and the same acceleration along the straight line as P has along the arc. Then for the motion of Q we have $a = -Ax$, or the motion of Q is S. H. M. The period of vibration of Q, of P, and of the rigid body are evidently equal, say T. But the period of Q's motion is (§ 40)

$$T = 2\pi\sqrt{-\frac{x}{a}}$$

$$= 2\pi\sqrt{-\frac{r\theta}{r\alpha}}$$

$$= 2\pi\sqrt{-\frac{\theta}{\alpha}}. \qquad (2)$$

Hence the angular acceleration and angular displacement of the rigid body are connected by the relation

$$\alpha = -\left(\frac{2\pi}{T}\right)^2 \cdot \theta. \qquad (3)$$

If a body has an angular acceleration opposite to and proportional to its angular displacement, its motion is angular S. H. M., and the period of vibration can be found by means of (2) or (3).

117. Torsional Pendulum. — If a body attached to a wire be turned through an angle and released, it will perform

angular vibrations. The motion is due to the fact that the twisted wire, tending to untwist, exerts a couple on the body and so sets the body in rotation, and the body, when once started, tends to persist in its motion owing to its moment of inertia.

Let the length of the wire be l, and let the couple applied to the free end be C, the other end being fixed; then the angle, θ, through which the wire is twisted, is, by Hooke's Law of Elasticity (see § 57), proportional to C; it is also proportional to the length l, for each unit of length is twisted to the same amount. Hence $\theta \propto Cl$, or $C = \frac{\tau\theta}{l}$, where τ is a constant for the same wire, and is called its *constant of torsion*; τ may also be defined as the couple per unit length per unit angle required to twist the wire. The couple exerted by the twisted wire is equal and opposite to C or it equals $-\frac{\tau\theta}{l}$.

Let the torsional pendulum be displaced through an angle θ, and let the angular acceleration imparted to it by the wire be α, then (§ 71)

$$-\frac{\tau\theta}{l} = I\alpha$$

and
$$\alpha = -\frac{\tau}{Il}\theta,$$

I being the moment of inertia of the pendulum. Now for a given pendulum, $\frac{\tau}{Il}$ is a constant. Hence the motion is angular S. H. M., and the period of vibration is

$$T = 2\pi\sqrt{\frac{Il}{\tau}}.$$

118. Comparison of Moments of Inertia by the Torsional Pendulum. — It follows from the last section that if two bodies are hung in succession from the same wire, and if their respective moments of inertia are I and I_1, and their periods of torsional vibration T and T_1, then

$$\frac{I_1}{I} = \frac{T_1^2}{T^2}.$$

If the period is T when a body of moment of inertia I is attached, and T_1 when to this body is attached another of moment of inertia i, then

$$\frac{I+i}{I} = \frac{T_1^2}{T^2},$$

or

$$I = i \frac{T^2}{T_1^2 - T^2}.$$

This suggests an experimental method of finding the moment of inertia of a body, no matter how irregular it may be in form.

Exercise XXVIII. The Torsional Pendulum. Comparison of Moments of Inertia

The upper end of a vertical wire is held in a clamp, and the lower end is attached to an axis that passes through the centre of a right-angled block of wood and is perpendicular to one pair of faces of the block.

To fix the position of the block when it is at rest, fasten a pin in the under surface near an end, and adjust a support in which is an upright pin until the two pins are in line when the block is at rest. Find by means of a stop-watch, or by counting the ticks of a clock or chronometer, the time required for a number of oscillations of the pendulum. If a chronometer circuit (foot-note p. 86) is used, begin counting seconds after a silence of the relay, and note the nearest second and fifth of a second at which the block passes its position of

rest; count the oscillations until the next silence of the relay, and again note the time, to a fifth of a second, when the block passes through its position of rest. The observations of the time of vibration should be repeated a number of times and the average taken. Then place two equal lead cylinders of known mass on the block so that the centres of the cylinders are at equal distances from the axis of oscillation, and find the time of oscillation as before.

Fig. 62.

From these observations, together with the masses and radii of the lead cylinders and their distances from the axis of oscillation, the moment of inertia of the block can be calculated (§ 118). The moment of inertia should also be found by direct calculation from the formula proven in § 74.

The constant of torsion of the wire can be readily deduced from the above results, and then the moment of inertia of any other body can be found by attaching it to the same wire and finding the time of vibration. This method may be applied to find the moment of inertia of a circular cylinder about an axis at right angles to its geometrical axis. This will afford a test of the formula proven in § 82. Or the moment of inertia of a sphere of wood may be found and the formula $I = \frac{2}{5} MR^2$ tested.

DISCUSSION

(*a*) Meaning and proof of formula used.

(*b*) Effect of errors in adjustment:

1. If the line of the wire be 2 mm. from the centre of the block.
2. If the block be inclined at 2° to the horizontal.

(*c*) How much error would result from supposing that the lead cylinders acted as if concentrated at their centres?

(*d*) In the case of an ordinary pendulum the arc of vibration must be small. Need this be so in the case of a torsional pendulum?

(*e*) Might a bifilar suspension (two parallel vertical cords) be used instead of a wire, without any change in the calculation?

119. The Compound Pendulum. — A rigid body vibrating under the influence of gravity about a fixed horizontal axis is called a *compound* or *physical pendulum*. Let S be the projection of the axis of suspension on a vertical plane through the centre of gravity C. Let SC be denoted by h, and let the pendulum be displaced through an angle θ; then the perpendicular from S on the vertical line through C is equal to $h \sin \theta$. Hence, if mg is the vertical force of gravity acting at the centre of gravity of the body, the moment of gravity about S is $-mgh \sin \theta$, negative when θ is positive, and *vice versa*. If I is the moment of inertia of the pendulum about the axis S, and α its angular acceleration,

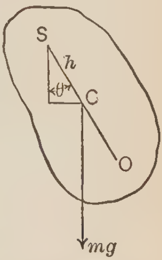

Fig. 63.

$$-mgh \sin \theta = I\alpha,$$

or
$$\alpha = -\left(\frac{mgh}{I}\right)\theta,$$

if θ be a small angle. Hence, for small angles of vibration the motion is angular S. H. M., and the period of vibration is (§ 116)

$$T = 2\pi\sqrt{\frac{I}{mgh}}.$$

If the radius of gyration about an axis through the centre of gravity, parallel to the axis of suspension, is k_0,

$$I = m(k_0{}^2 + h^2),$$

and
$$T = 2\pi\sqrt{\frac{k_0{}^2 + h^2}{gh}}.$$

(*a*) *Equivalent Simple Pendulum.* — If the above formula be compared with the formula for the time of vibra-

tion of a simple pendulum (§ 44), it will be seen that the compound pendulum vibrates in the same time as a simple pendulum whose length is

$$l = \frac{k_0^2 + h^2}{h}.$$

(*b*) *Centre of Oscillation.*—It follows from the above that the compound pendulum vibrates as if its whole mass were concentrated at a point O in SC such that

$$SO = l = \frac{(k_0^2 + h^2)}{h}.$$

The point O is called the *centre of oscillation*, corresponding to S, which is called the *centre of suspension*. Since $l > h$, S and O are on opposite sides of C.

(*c*) *Centre of Suspension and Oscillation Interchangeable.*—The equation for l may also be written

$$k_0^2 = h(l - h) = SC \cdot OC.$$

If the pendulum be suspended so as to vibrate about an axis through O, parallel to the original axis of suspension through S, then the new centre of oscillation, S', will be found by a similar equation, namely, $k_0^2 = OC \cdot S'C$. If these two equations be compared, it will be seen that S' must coincide with S. Hence a pendulum vibrates in the same time about an axis through any centre of suspension, and about a parallel axis through the corresponding axis of oscillation, or briefly, any centre of suspension and the corresponding centre of oscillation are interchangeable.

(*d*) *The Reversible Pendulum.*—A form of pendulum used for very accurate measurements of g is founded on the principle just stated. It consists of a rigid rod provided with two parallel axes of suspension in the form of

knife-edges. These axes are at right angles to the rod, on opposite sides of the centre of gravity, and in a plane passing through the centre of gravity. The position of the centre of gravity can be varied by one or two weights movable along the rod. If the positions of the weights be adjusted so that the times of vibration of the pendulum about the two axes are equal, then the length of the equivalent simple pendulum is the distance between the knife-edges (provided they be not equidistant from the centre of gravity — see below). Thus, as in the case of the simple pendulum, only two quantities, l and T, need be determined. (Poynting and Thomson, page 12.)

(*e*) *Parallel Axes about which the Times of Vibration are Equal.* — The radius of gyration about a certain axis through the centre of gravity being k_0, what is the distance from the centre of gravity of a parallel axis about which the pendulum vibrates as a simple pendulum of length l? To answer this we must find the value of h that will satisfy the equation $h^2 - hl + k_0^2 = 0$, l and k_0 being given. The solution is

$$h = \tfrac{1}{2}\left(l \pm \sqrt{l^2 - 4\,k_0^2}\right).$$

Hence, if l is greater than $2\,k_0$, there are two values of h that satisfy the conditions, and their sum is l. But nothing has been specified as to the direction in which h is to be measured from the centre of gravity. Hence all the parallel axes about which the pendulum vibrates in the same time pass through two circles, and the sum of the radii of the circles equals the length of the equivalent simple pendulum. But the length of the equivalent simple pendulum also equals the distance between a centre

of suspension and the corresponding centre of oscillation. Hence, as the centre of suspension travels around one of the circles, the centre of oscillation travels around the other.

(The reader should interpret the solution of $h^2 - lh + k_0^2 = 0$ when $l = 2k_0$, and when $l < 2k_0$.)

(*f*) *Curve of h and l.*—At various points along a line AB (Fig. 64), passing through the centre of gravity of the pendulum, suppose parallel axes of vibration fixed in the body perpendicular to AB. Let the length, l, of the equivalent simple pendulum corresponding to each one of these axes be determined experimentally. Then assume AB as an axis of abscissæ, and the centre of gravity as origin, and plot a curve with the values of h as abscissæ, and the corresponding values of l as ordinates. This curve will show at a glance the general relation between h and l. The form of this curve could have been predicted. For, corresponding to any value of l (above a certain limit $2k_0$), two values of h can be found on each side of the centre of gravity, the two smaller values being equal, and likewise the two larger values. Hence for any given value of l there are four points, P, Q, R, S, on the curve, and $PR = QS = l$.

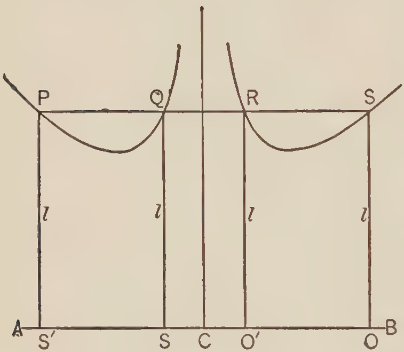

Fig. 64.

(*g*) *Graphical Method of solving* $k_0^2 = h(l - h)$.—On the line AB, referred to in (*f*), erect at the centre of

gravity a perpendicular, $CK = k_0$. With a pair of compasses find the point P on AB such that $KP = \frac{1}{2}l$, and, with P as centre, draw through K a circle cutting AB in S and O.

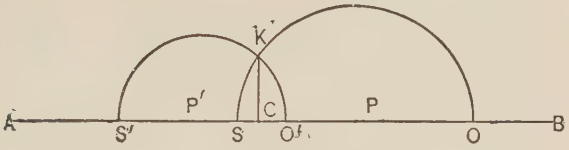

FIG. 65.

Since $CK^2 = CS \cdot CO$ and $CS + CO = l$, either CS or CO represents the required value of h. Two circles can be drawn according to the above directions. Hence we get four points on AB, about which the pendulum vibrates as a simple pendulum of length l. If this diagram be rotated about CK, S and C will describe the two circles referred to in (e).

By reversing the construction we can find k_0 if h and l (*i.e.* CS and CO) are known. All the circles like those in the diagram, drawn with known corresponding positions of C and S, should pass through K.

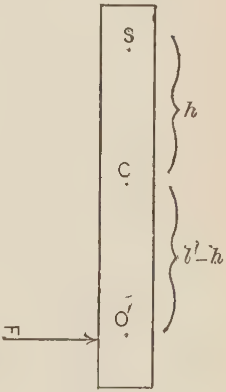

FIG. 66.

(*h*) *Centre of Percussion.*—If the pendulum be at rest, at what point must a horizontal force be applied to it so that it will start without exerting any side-force on the support? Here we are concerned only with horizontal forces and motions. Hence we may neglect gravity and suppose the body free in space. The

problem then is to find where F must be applied in order that when motion begins S may remain at rest as if fixed. The moment of F about S is Fl', if l' is the distance of O', the point of application of F, from S. Hence if k be the radius of gyration about S, the body will start rotating about S with an angular acceleration $\alpha = \frac{Fl'}{mk^2}$ (§ 71). But a force F applied to a free body of mass m starts the centre of mass with a linear acceleration $a = \frac{F}{m}$ (§ 85). Since S remains at rest, $a = h\alpha$ (§ 30).

Hence $$\frac{F}{m} = \frac{Fl'h}{mk^2},$$

or $$l' = \frac{k^2}{h} = \frac{k_0{}^2 + h^2}{h}.$$

Thus the point O, called the *centre of percussion*, coincides with the centre of oscillation.

Exercise XXIX. The Compound Pendulum

A simple form of compound pendulum that will suffice for the present purpose is a brass bar pierced by several holes and swinging on a knife-edge that passes through one of the holes. For finding the length of the equivalent simple pendulum a simple pendulum of adjustable length is hung from the same knife-edge.

The centre of gravity of the bar may be found by balancing it across the knife-edge. The point should be marked by a lead pencil. To prevent confusion one end of the bar may be lettered A and the other B, and the holes numbered from one end of the bar to the other. The distance, h, from the centre of gravity to that point on the circumference of each hole which will be in contact with the knife-edge should be carefully measured and the results tabulated.

Suspend the pendulum on the knife-edge and find the length l of the equivalent simple pendulum for each position of the axis of sus-

pension and tabulate the results. Similar observations may be made for axes outside of the bar by attaching a cord to the end of the bar and swinging the bar from the end of the cord.

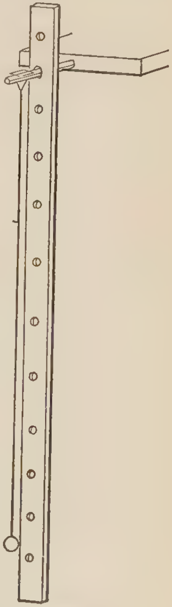

FIG. 67.

To represent these results graphically, draw a curve with values of h (positive toward A and negative toward B) as abscissæ and values of l as ordinates. Test the curve by measurements between the two branches as indicated in (f) and tabulate the results.

Some observations of the length of the equivalent simple pendulum for axes that do not intersect the axis of the bar may be made by tying the ends of a cord to holes in the bar and hanging the cord over the knife-edge. Values of l should be obtained for three or four such positions of the axis of suspension and the statements in (e) verified. Finally k_0 should be deduced graphically by the method suggested in (g).

DISCUSSION

(a) Meaning and proof of formulæ.

(b) Minimum length of equivalent simple pendulum.

(c) Position of axes about which the time of vibration is a minimum.

(d) Use of a compound pendulum for accurate determinations of g.

(e) Where should a base ball strike the bat to cause no jar on the hands?

120. Effect of an Angular Velocity about One Axis and an Angular Acceleration about an Axis at Right Angles to the First. The Gyroscope. — We have already had several illustrations of an analogy between the motion of translation of a particle and the motion of rotation of a rigid body (§§ 30, 71, 116). This analogy between rotation and translation extends still

further and we shall consider one interesting example. The motion of a particle that has a constant speed and a constant acceleration at right angles to the speed is a uniform circular motion (§ 33). What is the effect of a constant angular speed about one axis and a constant angular acceleration about another axis at right angles to the first? By analogy we should expect the result to be a constant revolution of the axis of rotation.

This is the problem of the Gyroscope which in its simplest form consists of a heavy wheel (of mass m) rotating with angular speed (ω) about a horizontal axis (OC) and supported at a point (O) in the axis of rotation. A full discussion of this instructive and interesting apparatus would be far beyond the scope of this book. The following brief and incomplete account will at least suggest the chief characteristics of the motion.

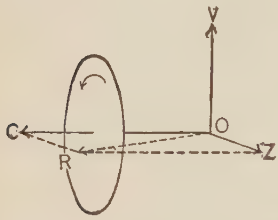

Fig. 68.

Let us first suppose that the wheel, while rotating about its axis OC with a constant angular velocity ω, is kept revolving about the axis OV with an angular velocity ω' and inquire what moment of force is necessary to keep up the revolution about OV. That some moment of force is necessary is evident, for, while the angular momentum about OV is constant in both magnitude and direction, the angular momentum $I\omega$ about OC is constant in magnitude but changes steadily in direction as OC revolves. (Compare the change of *direction* of momentum as a particle revolves in a circle.) After a short time t, OC will have turned into a position OR where the angle $COR = \omega' t$.

If the angular momentum of the wheel be represented by OC in its first position and by OR in its position after time t, OC and OR will be equal in length and, if the parallelogram $OCRZ$ be completed, OZ will represent the change of angular momentum. (The angular momentum about OV, being constant, suffers no change.) This change will require a moment of force, say L, about OZ, and, since the change is produced in time t, the angular momentum represented by OZ must equal Lt. Hence OZ and OC are proportional to Lt and $I\omega$ respectively. Now the small angle COR or $\omega' t$ may be taken as equal to $CR \div OC$ or $OZ \div OC$.

$$\therefore\ \omega' t = \frac{Lt}{I\omega}$$

or
$$L = I\omega\omega'.$$

Hence for steady revolution about OV the wheel must be acted on by a force that has no moment about OV or OC but has a moment $I\omega\omega'$ about an axis OZ always at right angles to OV and OC. (Compare the formula for centrifugal force, § 64, written in the form $F = mv\omega'$ where ω' or $v \div r$ is the rate at which the direction of mv rotates.) Now the weight of the wheel acting at its centre of mass at a distance h from O supplies precisely such a moment, mgh, about OZ. Hence if the wheel be started with an angular velocity $\frac{mgh}{I\omega}$ about OV it will, under the action of gravity, continue to revolve about OV with that angular velocity. Such a motion is called *precession*. If the rotating wheel be merely released without any angular velocity about OV being imparted to it, gravity will at

first cause a fall of the centre of gravity, but, since this will be accompanied by angular momentum about OZ, precession will set in and will continue at a rate given by the above formula but oscillations similar to those of a badly thrown quoit will accompany the precession.

The necessity for an initial impulse about OV may also be stated in the following way. Precession about OV implies an angular momentum about OV. If this be not supplied, the start will be opposed by inertia in the form of moment of inertia of the wheel about OV. The effect of inertia will be equivalent to that of an opposing moment of force about an axis OV' drawn vertically downwards, and the effect of this will be to depress the axis OC. The same effect follows any attempt to oppose the precession of the wheel when once started, whereas an attempt to accelerate the precession causes an elevation of OC.

If T be the period of precession, *i.e.* the time of one complete revolution about OV, then

$$T\omega' = 2\pi.$$

Hence
$$T = 2\pi \frac{I\omega}{mgh}.$$

The rotating armature of a dynamo on a ship that is rolling or pitching acts like a gyroscope. If the axis of the armature be at right angles to the length of the ship and the ship be rolling with an angular velocity ω' while the armature of moment of inertia I is revolving with angular velocity ω, then the bearings must supply the horizontal couple $I\omega\omega'$. If the distance between the bearings be a, the *horizontal* pressure on each bearing will be $\frac{I\omega\omega'}{a}$.

Exercise XXX. The Gyroscope

Apparatus. — A steel rod is attached by a double nut to the axle of a bicycle wheel; this rod passes loosely through a hole in the vertical steel axis used in previous exercises, being carried by a pin that

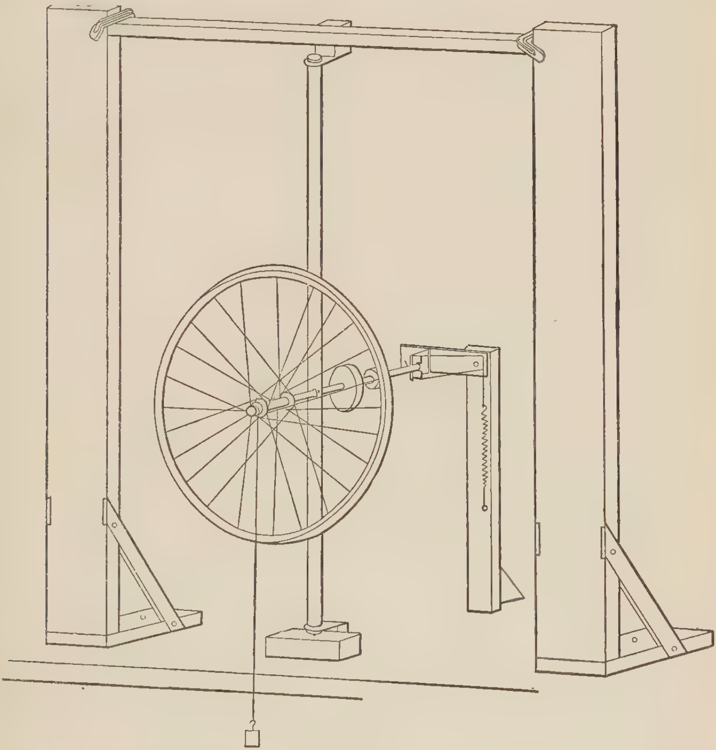

FIG. 69.

passes through rod and axis. The pin fits tightly into the rod and its ends, which are ground to knife-edges, rest loosely in the holes in the vertical axis so that the rod is free to vibrate in a vertical plane. A weight to counterpoise the wheel can be clamped on the rod and a smaller movable weight can be clamped at any desired point on

the rod so as to produce a moment of force about the line of the knife-edges of the pin. The wheel is started into rotation about its own axis by means of a thread that is wrapped around the hub of the wheel and carries a weight, the end of the rod being meanwhile held. The necessary initial impulse may be given with the hand, but a more satisfactory means is the simple starting device shown in the accompanying figure. A hinge turning about a vertical axis is attached by a cord to a spring the tension in which can be varied. The hinge being in the position shown in the figure, a loop of wire attached to the hinge encircles the end of the rod. If the hinge be released the spring will rotate it and start the gyroscope with an impulse that depends on the tension of the spring. The hinge can be clamped by means of a hook and released by a jerk on a cord attached to the hook. When drawn aside by the spring the hinge will not be in the way of the rod as it returns after a precession of the gyroscope. The block on which the hinge is mounted can be turned through 180° in order to start the gyroscope precessing in the opposite direction.

While the counterpoise and the small weight afford a means of greatly varying the action of the gyroscope, it will be well in making a first quantitative test to discard both and use the instrument in its simplest form. What immediately follows applies to this case.

Calculation of T. — To calculate the period of precession, the product, $I\omega$, and the moment mgh, must be known. The angular momentum, $I\omega$, of the wheel can be calculated from the mass, m', of the descending weight, the radius, r, of the hub, and the time of descent; for if α be the angular acceleration of the wheel,

$$m'gr = I\alpha,$$

and if t be the time during which the thread is attached to the wheel,

$$m'grt = I\alpha t = I\omega.$$

This is on the assumption that the frictional resistance to the rotation of the wheel is negligible. This resistance is small, but a rough estimate of it can be made by finding what small weight m'' attached to the thread will keep the wheel in steady rotation. Before $I\omega$ is calculated, m'' must be subtracted from m'.

The time of descent of m' should be ascertained with all the care possible. This should be done several times, m' being always started

from the same height. The length of the thread should be such that m' will reach the floor as nearly as possible at the moment when the thread becomes detached from the wheel. The wheel should be released exactly on a tick of the clock. The succeeding ticks should be counted, and the time at which m' strikes the floor estimated to one-fifth of a second. A strong effort should be made to have the separate determinations as independent as possible.

The value of mgh cannot be calculated directly since h is not readily measured. A simple and accurate method is to hang a weight from the axis by a thread with a loop that slips on the axis and adjust the position of the thread until the wheel is counterbalanced. The moment of the weight is readily found and that of the wheel is equal to it but with the opposite sign. For accuracy the distance of the thread from the knife-edge should be large.

Observation of T. — The period of precession should next be observed several times with similar care. The mean should agree with the calculated value within a fraction of a second.

The above method should also be applied to one or two cases with the counterpoise and small weight on the axis. The effect of attempting to accelerate or retard precession or tilt the axis, and also the nature of the oscillation that accompany precession as well as other points suggested by § 119 should be examined.

DISCUSSION

(*a*) Why is the axis of the wheel gradually depressed as precession continues?

(*b*) What effect does the frictional retarding force of the air and of the bearings on the wheel produce?

(*c*) What determines the direction of precession?

(*d*) Is the period of precession influenced in any way by the tension of the spring and the strength of the initial impulse it produces?

(*e*) Why is the impulse required of the spring much less when the counterpoise is discarded?

(*f*) Does gravity do any work during the motion?

(*g*) Does the impulse due to the spring in any way determine the period of oscillation of the gyroscope?

(*h*) Explain the motion of a top.

(*i*) Precessional motion of the earth. (Young's "General Astronomy," §§ 205–212.)

(*j*) A method of finding I experimentally by swinging the wheel as a compound pendulum.

(*k*) A method of finding ω (I being found separately) from the mass of m and its distance of descent.

(*l*) Why does the force of gravity not change the angular velocity of the body about OC?

(*m*) If the wheel be not quite symmetrical about OC, what will be the nature of the motion?

(*n*) The armature of a dynamo on a ship weighs 500 kilos. Its axis is at right angles to the length of the ship, and the radius of gyration is 50 cm. If the armature is running at a speed of 500 revolutions per minute, and the distance between the centres of the bearings is 50 cm., what is the pressure on the bearings when the ship is rolling at the rate of $\frac{1}{5}$ of a radian per second?

REFERENCES

Mach's "Science of Mechanics," Chapter II.
Perry's "Spinning Tops."
Worthington's "Dynamics of Rotation."

ELASTIC SOLIDS AND FLUIDS

CHAPTER X

MECHANICS OF ELASTIC SOLIDS

121. Solids and Fluids. — A *solid* is a body that has a definite shape even while it is acted on by forces which tend to produce a change of shape. A *fluid* is a body that continues to change its shape so long as it is subjected to forces which tend to produce a change of shape.

The definition of a solid does not imply that the shape of a solid is invariable. Some change of shape always occurs, when deforming forces are applied to a solid; but the solid assumes a new shape, which it maintains so long as the forces remain constant. (Some slight qualification of this last statement is necessary, but, for convenience, we shall postpone it.)

122. Strain. — Any change of shape or volume or change of both shape and volume is called a *strain*. A change of shape without any change of volume is called a *shear*. For example, when a rod is twisted through a small angle a small part of the rod that was originally cubical assumes a new shape, but suffers no change of volume. A change of volume without any change of shape has usually not received any special name; but, to

avoid the repetition of the phrase "change of volume without change of shape," we shall call it a *squeeze*. It is illustrated by the compression of a cube into a smaller cube or a sphere into a smaller sphere. A dilatation is a negative squeeze.

Shears and squeezes are called *simple strains*. Strains which involve changes of both shape and volume, *e.g.* the strain of a stretched wire or a bent beam, can be resolved into shears and squeezes.

Any strain which is of the same kind and magnitude at all points throughout the strained body, is called *homogeneous strain*. All parts of a stretched uniform wire are similarly affected and the strain is homogeneous. When a rod is twisted, the strain is greater near the surface than near the axis and the strain is non-homogeneous. The same is true of a bent beam. In such a case the strain in a very small part of the body may be regarded as homogeneous.

123. Numerical Measure of a Squeeze. — When a squeeze is homogeneous, it is measured by the proportion in which the whole or any part of the body changes in volume, or, if v_1 be the original volume of any part of the body and v_2 the volume to which that part is reduced, the measure of the squeeze is $(v_1 - v_2) \div v_1$.

If the squeeze is not homogeneous, to find its measure at any point we must suppose v_1 to be the volume of a small part surrounding that point, and the measure of the squeeze at that point is the value approached by $(v_1 - v_2) \div v_1$ as the part considered is taken smaller and smaller without limit.

124. Numerical Measure of a Shear. — First consider how a shear may be produced. Between two horizontal boards place a large cube of firm jelly (a calf's foot jelly containing some glycerine will do admirably). Give the upper board a horizontal displacement in a direction parallel to one vertical face $ABCD$ of the cube. Any plane parallel to $ABCD$ is called a *plane of shear*. Each square in a plane of shear becomes a rhombus. All planes in the body parallel to the boards move parallel to the boards, and the measure of the shear is the relative displacement of any two of these parallel planes divided by the distance between them. In the figure the shear is $Bb \div AB$. If ϕ is the amount by which a right angle in a plane of shear changes $\tan \phi = Bb \div AB =$ the shear, and if the shear is small, ϕ may be taken as its measure instead of $\tan \phi$.

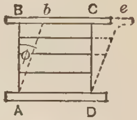

FIG. 70.

125. Elasticity. — The property that enables a body to recover from strain is called *elasticity*. When the strain recovered from is a shear, the elasticity is called *elasticity of form;* when it is a squeeze, the elasticity is called *elasticity of volume*. Forms of elasticity corresponding to the various forms of compound strain have not, in general, received special names.

A body that recovers completely from any form of strain is said to have *perfect elasticity* of that form; if the recovery is incomplete, the elasticity is said to be imperfect. Probably no solid has, in reality, perfect elasticity of any kind; but many solids are so nearly perfectly elastic, when the strain does not exceed a certain amount

called *the elastic limit*, that they may for many practical purposes be regarded as perfectly elastic.

A solid, such as putty or lead, that has a very small elastic limit when sheared is said to be *plastic;* a permanent change of form of such a body can be produced by comparatively small forces.

126. Stress. — When an elastic body is in a state of strain, there are internal forces, actions and reactions, between contiguous parts of the body. As a somewhat rough illustration consider the state of a book when pressures are applied normally to the covers. Any leaf, A, presses against a contiguous leaf, B, with a certain force, and B presses back against A with an equal and opposite force. If the forces applied to the cover be tangential instead of normal, the leaf A will exert a tangential force on the leaf B, and B will exert an equal and opposite tangential force on A. The condition of a pillar that supports a weight is somewhat similar to that of the book in the first case, and the condition of any small part of a twisted rod is similar to that of the book in the second case.

The action and reaction between the contiguous parts of a strained body constitute a *stress*. The measure of the stress is the magnitude of the force (action or reaction) per unit of area. In some cases the stress is equal to the external force, per unit of area, applied to the body. When a solid is immersed in a liquid to which pressure is applied, the pressure per unit area within the solid equals the pressure, per unit of area, exerted on the solid by the liquid. When a wire is stretched by a weight

attached to it, the tension per unit area of cross-section of the wire equals the weight sustained divided by the cross-section.

In many cases the stress cannot be measured directly by the external force. This applies to a beam that is bent by a weight; if the weight is increased, the stress at any particular point is increased in the same proportion; but the stress is different at different points, and so it cannot be measured by the magnitude of the weight. There may be a stress within a body to which no external force is applied. An iron casting, when cool, has internal strains and stresses even when no external force acts on it, and a glass vessel when heated irregularly may break, owing to the magnitude of the internal strains and stresses.

127. Hooke's Law. — Careful experiments have shown that, so long as the strain in an elastic body is within the elastic limit, *the ratio of the measure of the stress to that of the strain is constant.* This law was first stated by Robert Hooke (in 1676) in the words *ut tensio sic vis*, or, *stress is proportional to strain.* Hooke illustrated this law by the stretching of a spiral spring, the twisting of a wire, the bending of a plank, etc. Some of these we shall consider later.

Striking evidence of the correctness of Hooke's Law is afforded by the vibrations of an elastic body such as a tuning-fork. The frequency of a S. H. M. depends on the ratio of the restoring force to the displacement (§ 57), and if this ratio changed, the frequency would change, and the pitch of the note given by the tuning-fork would also change. But the pitch of a tuning-fork remains constant although the amplitude of the vibrations decreases. Now, the displacement of

a point on the fork is proportional to the strain, and the restoring force is proportional to the stress, and the fact that the pitch remains constant as the vibrations die away shows that the fork obeys Hooke's Law. A tuning-fork of definite pitch may be made of any ordinary metal, even lead, and this shows that, within the elastic limit, all ordinary metals obey Hooke's Law with great, if not perfect, accuracy.

128. Moduli of Elasticity. — The *modulus*, or measure, of the elasticity of a body corresponding to any form of strain, is *the ratio of the measure of the stress to that of the strain.* There is, therefore, a modulus for each possible form of strain, but only a small number need be specially considered. Two that may be considered as the *principal moduli* are the *shear modulus* (also called the *simple rigidity*) and the *bulk modulus.*

The *bulk modulus* is the ratio of the squeezing stress to the squeeze. The measure of the squeezing stress is the pressure per unit of area, p, within the body. If the squeeze is due to liquid pressure applied to the body, the squeezing stress also equals the pressure per unit of area, p, in the liquid. The measure of the squeeze (§ 123) is $(v_1 - v_2) \div v_1$. Hence, denoting the bulk modulus by k,

$$k = \frac{p}{\frac{v_1 - v_2}{v_1}} = \frac{pv_1}{v_1 - v_2}.$$

The reciprocal of k is called the *coefficient of compressibility* of the substance. It evidently equals the proportion in which the volume is decreased when unit pressure is applied to the body.

The *shear modulus* is the ratio of the shearing stress

to the shear. The shearing stress is measured by the tangential force, T, per unit of area within the body. Denoting the measure of the shear by ϕ (§ 124) and the shear modulus by n,

$$n = \frac{T}{\phi}.$$

129. Torsion of a Wire or Rod. — The strain at any point of a wire or rod subjected to a slight twist is a shear. Consider a very small cube one edge of which is parallel to the length of the wire, while a second edge lies along a radius of the section of the wire, and a third is part of the circumference of a circle coaxial with the wire. A consideration of Fig. 71 will show that the strain of the cube is similar to that of the cube of jelly referred to in § 124. The stress is also a shearing stress. The shear and the shearing stress may be found by considering the dimensions of the wire and the twist it undergoes when a known couple is applied to one end, the other end being clamped.

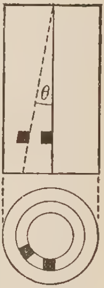

Fig. 71.

The relations between the twist of a wire, the dimensions of the wire, and the couple that produces the twist can be found by experiment or calculated by theory. The following exercise will, in a rough way, illustrate the experimental method.

Exercise XXXI. The Torsion of a Wire

A vertical wire is clamped at both ends and carries a horizontal disk which is clamped to the middle of the wire. The recording disk of Exercise XIV may be used for the purpose, the aperture being reduced by the insertion of a "connector" (such as is used in elec-

trical circuits), the set screw of which will serve to clamp the wire. A protractor, or graduated paper circle, is fastened to the upper side of the disk, and a bent wire attached to a cross-bar serves as an index in measuring the rotation of the centre of the wire. Tangential forces are applied to the disk by means of threads which pass over pulleys,

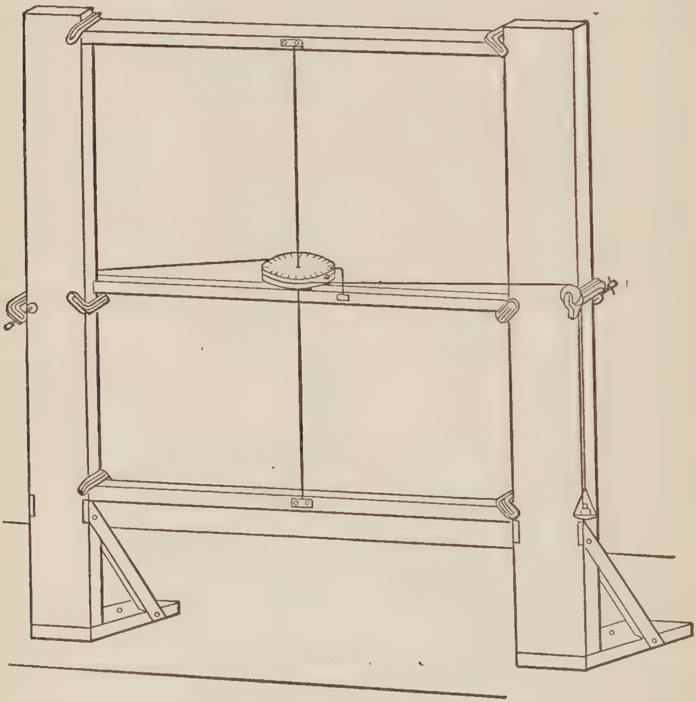

Fig. 72.

attached to the framework, and carry scale pans and weights. (Instead of the scale pans, weights, and pulleys, calibrated springs may be used, though not as readily.)

We may first inquire whether the angle, θ, through which the middle of the wire is twisted is proportional to some power, p, of the couple, L, applied to the wire. If so, $\theta = c \cdot L^p$, c being some constant

that does not change as L and θ vary. Hence $\log \theta = \log c + p \log L$. If couples $L_1, L_2, L_3, \cdots$ produce twists $\theta_1, \theta_2, \theta_3, \cdots$, and if the values of $\log L_1, \log L_2, \cdots$ plotted as ordinates against the value of $\log \theta_1$, $\log \theta_2, \cdots$ as abscissæ give a straight line, it will show that θ is proportional to some power of L. From the values of L_1, L_2, and θ_1, θ_2 a value of p may be obtained by substituting in the equation last stated and eliminating $\log c$.

$$p = \frac{\log \theta_1 - \log \theta_2}{\log L_1 - \log L_2}.$$

Other values of p may be obtained from pairs of corresponding values of L and θ. These values of p should agree as well as could be expected, when the unavoidable *errors of observation* are considered.

In a similar way we may find the relation between the twist of the wire and its length, L being kept constant. The length may be changed by altering the position of the bars to which it is clamped.

Finally, by using three or more wires of the same material and length, twisted by the same couple, we may find the relation between twist and radius. In this case, unless carefully selected wires are chosen, the results will probably not be so satisfactory as in the preceding cases. In fact, the results may be considerably affected by a serious *error of method* (in addition to the *errors of observation*); namely, the use of wires that do not consist of the same material *in the same state*. It is well known that the process of wire drawing has considerable effect on the physical state of the material. Divergences between the results amounting to several per cent may be found; but, as their causes are understood, their existence need cause no dissatisfaction with the results.

Allowing for errors both of observation and of method, the results will show that

$$\theta \propto \frac{Ll}{r^4}.$$

DISCUSSION

(*a*) Do the observations confirm Hooke's Law?

(*b*) Describe the nature of the strain in the wire.

(*c*) Does the magnitude of the strain at a point depend on the distance of the point from the axis?

(*d*) Is the strain everywhere the same at equal distances from the axis?

(*e*) Where will fracture begin if a uniform glass rod be twisted too much?

(*f*) From the twist of the free end calculate the twist of any other cross-section.

(*g*) Did the tension on the wire affect the experimental results?

(*h*) What is the nature of the strain in a stretched spiral spring?

(*i*) What is the magnitude of the shear of a cubical part of the wire (§ 129), if the side of the cube is 0.01 mm., the length of the wire 1 m., and its radius 1 mm., supposing the centre of the cube halfway from the axis to the surface of the wire and the twist of the whole wire 30°?

130. Theory of Torsion of a Uniform Wire. — For definiteness we shall suppose that the ends of the wire are sections perpendicular to the axis and are cemented firmly to disks, one disk being held fixed while the other is turned about the axis of the wire through an angle θ. Let the length of the wire be l and its radius R. Consider two normal sections unit distance apart. One is turned through an angle $\theta \div l$ relatively to the other. Hence at a distance r from the axis the measure of the shear is $r\theta \div l$, and that of the shearing stress $nr\theta \div l$.

Suppose the area of the end which is attached to the rotated disk to be divided up into a large number of small areas $s_1, s_2, \cdots$, their respective distances from the axis being $r_1, r_2, \cdots$. To these the disk applies tangential forces $nr_1s_1\theta \div l$, $nr_2s_2\theta \div l, \cdots$. The tangential force applied to each small area is perpendicular to that radius of the end section that passes through the centre of the area. Hence the sum of the moments of these forces about the axis, that is, the couple applied to the free end, is

$$C = \frac{nr_1^2s_1\theta}{l} + \frac{nr_2^2s_2\theta}{l} + \cdots = \frac{n\theta}{l}\Sigma sr^2.$$

Now Σsr^2 is the moment of inertia of a disk of the same form as the end section and of unit mass per unit area (it is often called the moment of inertia of the section). Denoting it by I,

$$C = \frac{n\theta I}{l}.$$

For a circular wire $I = \frac{1}{2}\pi R^2 \cdot R^2 = \frac{1}{2}\pi R^4$ (§ 75), and therefore

$$C = \frac{\pi n \theta R^4}{2l}.$$

To get the constant of torsion of the wire or the couple per unit length per unit angle required to twist the wire (§ 117), we put $\theta = 1$ and $l = 1$. Hence

$$\tau = \frac{1}{2}\pi n R^4.$$

(From the results of the last exercise calculate n.)

131. Kinetic Method of finding n. — The last formula of the preceding suggests a method of measuring the shear modulus of a wire. To the wire a body of known moment of inertia is attached and the time of a torsional vibration is observed (§ 117 and Exercise XXVIII). This gives the value of τ, and that of n is found from τ and R. (Calculate from the results of Exercise XXVIII the constant of torsion and the shear modulus of the wire.)

132. Stretch Modulus. — When a wire of length l is stretched to a length $l + x$, the measure of the strain is $\frac{x}{l}$, or the proportion in which the length is increased. If the whole force applied to stretch the wire is F, and the area of cross-section of the wire is s, the measure of the stress, or the tension per unit of cross-section, is $\frac{F}{s}$. Hence the stretch modulus is $\frac{F}{s} \div \frac{x}{l}$ or $\frac{Fl}{sx}$. The stretch modulus is

also called *Young's modulus* from the name of the physicist, Thomas Young, who first defined it (1807).

The definition of the stretch modulus suggests a direct method of measuring it. The length and radius of the wire are carefully measured, and then the stretch produced by hanging a known weight to the wire is carefully observed.

When a rod is shortened by longitudinal pressure, the strain is a negative stretch. The stress is in this case a *thrust* which may be considered as a negative stretching stress. The value of Young's modulus obtained from the strain and stress in the compressed rod is not appreciably different from the value found when the strain is a positive stretch.

133. Poisson's Ratio. — The stretching of a wire involves a change of shape, or a shear, since the length increases and the diameter decreases. Whether a change of volume also occurs can be found by careful measurements of the changes of length and diameter.

Let the initial radius be r_0 and the final r, and let l_0 and l be the corresponding lengths. If no change of volume took place, $\pi r_0^2 l_0$ would equal $\pi r^2 l$, or r and l would be connected by the relation

$$\frac{r}{r_0} = \left(\frac{l_0}{l}\right)^{\frac{1}{2}}.$$

Experiment shows that for no substance is this relation true, but that in all cases

$$\frac{r}{r_0} = \left(\frac{l_0}{l}\right)^{q}, \tag{1}$$

q being a constant for each substance. The smaller q is, the smaller is the ratio in which the radius decreases for a

given stretch; and since it is found that in all cases q is less than $\frac{1}{2}$ (see Table in Appendix), a stretch is always accompanied by an increase of volume.

The constant q is called *Poisson's ratio.* To explain why it is called a ratio let us suppose that the stretch is very small, so that $l = l_0 + x$, and $r = r_0 - z$, x and z being very small. Substituting in (1) and expanding,

$$\frac{r_0 - z}{r_0} = \left(\frac{l_0}{l_0 + x}\right)^q,$$

or

$$1 - \frac{z}{r_0} = \left(1 + \frac{x}{l_0}\right)^{-q} = 1 - q\frac{x}{l_0},$$

squares and higher powers of $x \div l_0$ being neglected.

Hence

$$q = \frac{z}{r_0} \div \frac{x}{l_0}. \tag{2}$$

Thus q may be defined as the ratio of the fractional decrease of radius to the fractional increase of length, both being supposed indefinitely small.

Exercise XXXII. Young's Modulus and Poisson's Ratio

A rubber cord about 3 mm. in diameter is clamped in the slitted end of a long vertical screw which turns in a nut attached to a horizontal bar. Any desired number of turns can be given to the screw by turning a lever on the top of the screw.

Through the cord two fine sewing needles are thrust, and the distance between them is read on a vertical mirror scale.

The mean diameter of the cord is found by observing the movements of two silk threads that wind around the cord as the screw is turned. Each thread is attached to the cord by a small loop that passes over one end of the upper needle. The direction of the thread is at first horizontal, but at a distance of 1 cm. from the cord it passes through a small screw-eye and hangs vertically in front of the mirror

scale. A small bullet suspended from the thread keeps the thread under definite tension and also serves as an index. The movement of the bullet can be read to about .2 mm. The use of two threads eliminates errors which would enter if the cord were drawn sidewise by the tension of a single thread. Allowance must be made for the slope of the thread on the rubber cord. The angle of slope can be calculated from the pitch of the screw. The screw should be given three or four turns before readings are begun.

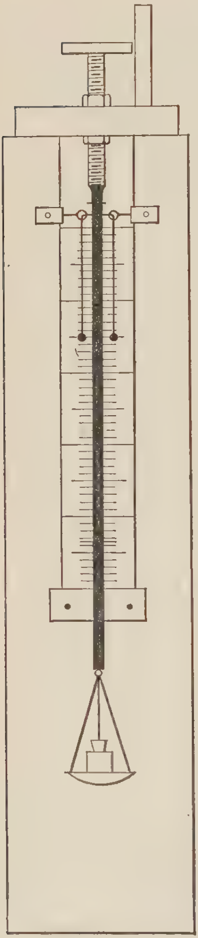

Fig. 73.

About 200 g. are placed in a scale pan attached to the cord. The positions of both ends of each needle are observed on the scale and also the positions of the lowest part of each bullet. The screw is then given ten or more complete turns, and all of the above readings repeated. The screw is next turned back to its initial position and the readings again repeated. The load is then increased by 100 g. and the above operations are repeated to obtain the new length and diameter.

From these readings Young's modulus can be calculated. For the value of the cross-section, the mean of its values before and after the addition of the 100 g. should be taken.

To obtain Poisson's ratio, take the logarithms of both sides of (1) of § 133.

$$\log r_0 - \log r = q(\log l - \log l_0).$$

Both constants should be obtained several times with different loads on the pan, increasing by steps of 100 g. until the rubber shows a decided permanent set. In calculating Young's modulus, use for the cross-section the mean of its values before and

after the last addition of 100 g. In calculating Poisson's ratio, take for r_0 and l_0 their values before the first 100 g. were added.

DISCUSSION

(*a*) Do the observations confirm Hooke's Law?

(*b*) Meaning of Young's modulus and Poisson's ratio.

(*c*) Why would an incorrect value for q be obtained if it were calculated from the experimental results with the aid of equation (2) of § 133? Would this apply to measurements of a metallic wire?

(*d*) What do your results show as regards changes of volume of rubber when stretched?

(*e*) Do the results indicate anything as regards the elastic limit of rubber?

134. Flexure of a Uniform Bar. — When a uniform bar is bent, longitudinal lines of particles on the convex side are lengthened, while those on the concave side are shortened. Lines on a certain intermediate surface, called the *neutral surface*, are not changed in length. A plane containing any one of these curved lines is called a *plane of bending*. If the bending is slight, the extensions and compressions are similar to those of a rod which is stretched or shortened by a longitudinal force, as in the measurement of Young's modulus.

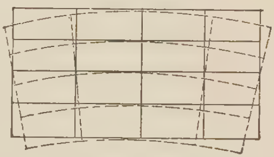

FIG. 74.

The relations between the amount of bending, the dimensions of the bar, and the force applied to the bar may be found by experiment or calculated by theory. The following exercise will illustrate the experimental method.

Exercise XXXIII. Flexure of a Bar

Uniform brass bars of rectangular cross-section are supported in succession on two knife-edges and various weights are hung from the

centre of each bar. The depression is measured by a lever, one end of which rests on the bar by means of two needle points, while a point near that end is supported on a cross-bar of the framework by means of a single needle point. A needle in the farther end of the lever moves down along a vertical millimetre mirror scale as the centre of the brass bar is depressed. Thus the movement of the centre of the bar is read on a scale enlarged in proportion to the ratio of the arms of the lever.

To find how the depression, x, of the centre of the bar thus loaded depends on (1) the force, F, applied to the bar, (2) the length, l, of the bar, (3) the width, b, of the bar, and (4) the depth, d, of the bar, proceed as in Exercise XXXII. The method is so entirely similar

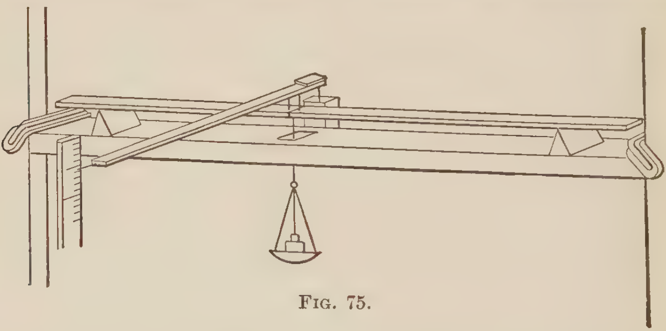

Fig. 75.

that it need not be restated. The actual value of x is not needed for the present purpose; the scale readings are proportional to the values of x and are sufficient. But for another purpose the actual values of x are required. Hence the lengths of the arms of the lever should be measured. The full length of the lever from the single needle point to the end of the index needle may be measured by an ordinary scale. The length of the short arm may be found by a micrometer caliper or by placing the lever on a finely divided scale.

(For this exercise bars should be supplied, three or more of which have the same thickness and length but different widths, while three or more have the same width but different thicknesses. Thus at least five bars will be needed. Bars of the same thickness may be sawn from the same sheet of brass.)

DISCUSSION

(*a*) Do the results agree with Hooke's Law?

(*b*) Regarding the bar as made up of parallel wires, how are the various wires changed in length?

(*c*) Would change in length of the wires account for the change of shape of the bar?

(*d*) Is there any change in the cross-section of the bar?

(*e*) What relative motion of two adjacent cross-sections takes place?

(*f*) What kind of vibrations would the loaded bar perform?

(*g*) It can be shown by mathematical methods that the depression of the bar is $\frac{Fl^3}{4\,Mbd^3}$. From this and the observations made calculate M.

(*h*) Deduce from the formula in (*g*) a formula for the depression produced by a weight attached to one end of a bar that is clamped horizontally at the other end.

135. Relation between Elastic Moduli. — Since the stretch of a wire and the flexure of a bar involve both shears and squeezes, it is evident that there must be a relation between the stretch modulus, M, the shear modulus, n, and the bulk modulus, k. This relation (the proof of which we must omit) is

$$M = \frac{9\,kn}{3\,k + n}.$$

The value of n for some substances, such as india-rubber, is very small compared with that of k, and M is, therefore, nearly equal to $3\,n$.

If M and n be carefully measured, k can be deduced. The direct measurement of k is difficult.

136. Potential Energy of Strain. — Work is done in straining an elastic body, and the strained body has an amount of potential energy equal to the work done. It can be

shown that the amount of this potential energy *per unit volume* of the strained body is *one-half the product of the stress by the strain.* As an example, consider a wire of cross-section s and length l initially under no tension, and suppose that it is stretched by a gradually increasing force to a length $l+x$, the force at this length being F. Since the force is, at each stage of the stretching, proportional to the extension (Hooke's Law), the diagram of work (§ 100) is a triangle, and the total work done or potential energy produced is, therefore, $\frac{1}{2}Fx$. If the amount of the stretch is small (as in the case of a metallic wire stretched within the elastic limit), the volume of the stretched wire is sl. Hence the potential energy per unit volume is $\frac{1}{2}\frac{F}{s}\cdot\frac{x}{l}$. Now, $\frac{F}{s}$ is the force per unit area, that is, the stress, and $\frac{x}{l}$ is the stretch per unit length, that is, the strain.

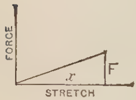

Fig. 76.

137. Imperfections of Elasticity. — So long as the strain of a body is within the elastic limit, a curve, plotted with stresses as ordinates and strains as abscissæ, is (at least very nearly) a straight line, as we have seen in the last three exercises. Further stress will cause the strain to increase more rapidly than the stress, and the curve will become concave downwards. Finally, a point, called *the yield point*, will be reached at which the strain will increase very rapidly and the material will cease to act as a solid and begin to *flow*. If the stress be then relaxed, a large permanent set will remain. The stamping of a coin is a striking illustration.

In many cases the elastic limit is somewhat indefinite, and the curve of stress against strain is everywhere concave downwards. When the stress is gradually decreased, the curve is not retraced, but another curve, concave upwards, is obtained. This is called *elastic hysteresis.*

Fig. 77.

In other cases (glass, for example), when a stress is produced and kept constant, the strain does not reach its full value at once, but continues to increase for some time. When the stress is relaxed, the strain nearly disappears, but a slight residual strain remains, which only slowly disappears. This is called *elastic lag.*

If an elastic body be in some way compelled to keep vibrating for a long time and be then left to vibrate freely, the vibrations will die away more rapidly than they would have if the body had not received such preliminary treatment. Lord Kelvin found that the torsional vibrations of a wire that had been kept in torsional vibration for a long time and was then set free died away to one-half in 44 or 45 vibrations, while the vibrations of a similar wire, started "fresh," took 100 vibrations to fall to one-half. This is called *fatigue of elasticity.*

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CHAPTER XI

MECHANICS OF FLUIDS

138. A fluid is a body that yields to the smallest deforming force; while there is any shearing stress, however small, in a fluid, it continues to flow; that is, the amount of shear continues to increase. In other words, *the shear modulus of a fluid is zero.*

Fluids are divided into *liquids* and *gases* according to the magnitude of their bulk moduli. The bulk modulus of a liquid is large, that is, it takes a large stress to produce a small strain; when the pressure on water is doubled, its volume is decreased by only one part in 20,000. The bulk modulus of a gas is small; when the pressure on a gas is doubled, its volume is decreased by one-half. The difference between liquids and gases seems to depend on the distances between particles, the particles of a liquid being comparatively close together while those of a gas are widely separated.

A liquid can have a definite "free surface," that is, a surface not confined by a solid or by another liquid; whereas a gas expands so as to occupy the largest space open to it.

139. Direction of Force on the Surface of a Fluid. — When a fluid is at rest, the resultant force exerted on its surface must be perpendicular to the surface; for if it were not,

it would have a component parallel to the surface, and this would cause a flow of the fluid. Against any force exerted on it a fluid exerts an equal and opposite reaction ; hence a fluid at rest presses perpendicularly against any surface in contact with it. These statements apply not only to the contact of a solid and a liquid, but also to that of two liquids, such as oil and water, which do not mix. But when a fluid is in motion, the force between it and the surface of a body in contact with it may be inclined to the surface ; a fluid flowing through a pipe tends to drag the pipe with it.

140. Pressure in a Fluid. — At any point in a fluid the part of the fluid on one side of an imaginary dividing plane through the point presses against the fluid on the other side. If the fluid on one side of the plane be supposed removed and a solid surface substituted, the latter will sustain the pressure of the remainder of the fluid.

The whole force of fluid pressure on any plane surface is called the *thrust* (or *total pressure*) on the surface, and the thrust on any plane surface divided by the area of the surface is called the *average pressure* on the surface. The value to which the average pressure on a small area surrounding a point approaches as the area is diminished is called the *pressure* (or pressure intensity) at that point ; this is otherwise expressed as the thrust per unit area at the point. If the pressure is the same at all points on a surface, it equals the thrust on any unit of area of the surface.

Pressure in a fluid is due either to the weight of the fluid, as in the case of water in a tank, or to force applied

to some part of the surface of the fluid, *e.g.* force applied to a piston that closes a cylinder containing the fluid. (The effect of attractions between particles of the fluid need not at present be considered, for the pressure thus caused does not act on a body immersed in or exposed to the fluid.)

141. Pressure at a Point. — In defining the pressure at a point it is not necessary to specify any direction, for at any particular point *the pressure has the same magnitude in all directions.* For, let O be the point and let A_1O and A_2O be any two directions through it.

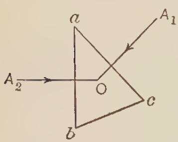

FIG. 78.

Around O describe a small triangular prism, two sides passing through ac and ab being at right angles to A_1O and A_2O respectively, while a third side passing through bc is equally inclined to A_1O and A_2O and the ends are parallel to the plane of A_1O and A_2O. The fluid within the prism is at rest. Hence the whole force on it in the direction bc is zero. But the only forces in this direction are components, in the direction bc, of the thrusts P_1 and P_2 on the surfaces through ab and ac respectively. Now P_1 and P_2 are equally inclined to bc, and, since their components in the direction bc must be equal and opposite, P_1 and P_2 must also be equal in magnitude. The faces through ab and ac are also equal in area. Hence the average pressures on these faces must be equal in magnitude. If the dimensions of the prism be supposed diminished without limit, the average pressures on ab and ac become the pressure at O in the direction A_1O and A_2O. Hence these are also equal. But A_1O and A_2O are any directions through

O. Hence the pressure at a point is the same in all directions.

In the above we have neglected the force of gravity on the prism of fluid; but since this is proportional to the volume of the fluid, that is, to the cube of the dimensions of the prism, while the thrusts on the faces are proportional to the squares of the dimensions, when the prism is reduced without limit the force of gravity vanishes in comparison with the thrusts.

142. Pressure at Different Points in a Fluid. — (1) Let A and B be two points in a horizontal line, and let AB be wholly in the fluid. Around AB as axis describe a cylinder with vertical ends. The fluid within the cylinder is at rest. Hence the resultant horizontal force on it is zero. Since its weight acts vertically and the thrusts on its sides are perpendicular to AB, the only forces in the direction AB are the thrusts on its ends, and these must therefore be equal in magnitude and opposite in direction. Hence the pressure per unit area at A must equal that at B. Thus the pressure is the same at all points in the same horizontal plane in the fluid.

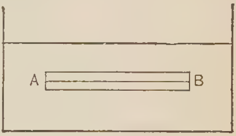

FIG. 79.

(2) Let A and B be two points in the same vertical line AB in the fluid. Around AB as axis describe a cylinder with horizontal ends. The fluid within the cylinder is at rest and the resultant vertical force on it is therefore zero. Therefore, if the thrust on the end A be P_1 and that on the end B be P_2, and if A be above B, $P_2 - P_1$ must equal the weight of the cylinder. If the density of

FIG. 80.

the fluid (or its mass per unit volume) be ρ, and if the length of the cylinder be h and its cross-section be a, the volume of the cylinder is ha, its mass is $ha\rho$, and its weight is $ha\rho g$.

$$\therefore\ P_2 - P_1 = ha\rho g$$

and
$$\frac{P_2}{a} - \frac{P_1}{a} = h\rho g.$$

The pressure on the end A is a uniform pressure and therefore the pressure, p_1, at A equals $P_1 \div a$. Similarly, the pressure, p_2, at B equals $P_2 \div a$.

Hence
$$p_2 - p_1 = h\rho g.$$

(3) Any two points A and B in the fluid can be connected by a broken line $ACD \cdots B$ consisting of horizontal and vertical steps. Along each horizontal step there will be no change of pressure and the total change of pressure along the vertical steps will be $h\rho g$, h being the difference of level of A and B.

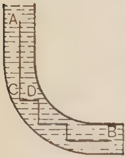

Fig. 81.

In the above we have assumed that the density ρ is the same at all points in the fluid. This is practically true for small bodies of fluid, but it is far from true for large bodies such as the atmosphere and the ocean.

The density of a gas is so small that, unless h be very large, $h\rho g$ is very small compared with p_1 or p_2. Hence in moderate volumes of a gas the pressure may be considered as everywhere the same.

143. Surface of Contact of Two Fluids. — The surface of contact of two fluids of different densities which are at rest and do not mix is horizontal. For, take two points

P and Q on the surface of contact, and let a vertical through P meet a horizontal through Q in a point A in the fluid of density ρ, while a vertical through Q meets a horizontal through P at the point B in the fluid of density ρ'. The pressures at A and Q are equal, and the pressures at P and B are equal. Hence the increase of pressure from A to P equals that from Q to B, or, denoting the common length of AP and QB by h, $h\rho g = h\rho' g$, and therefore $h(\rho - \rho') = 0$. Hence, since ρ and ρ' are unequal, h must be zero, or P and Q must be in a horizontal line.

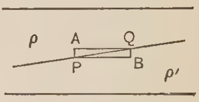

FIG. 82.

It follows from the above that the free surface of a liquid at rest is horizontal. This is also readily seen by considering that, if the surface were not horizontal, the vertical force of gravity would have a component parallel to the surface, and this would cause motion parallel to the surface.

If the pressure on the horizontal free surface of a liquid is P, the pressure p at a depth h below the surface is greater than P by $g\rho h$, or $p = P + g\rho h$.

144. Transmissibility of Fluid Pressure (Pascal's Principle). — In a fluid of constant density and at rest, the difference of pressure between two points depends only on the difference of level of the points and the density. Hence, *an increase of pressure at any point is accompanied by an equal increase at all points.* This is known as Pascal's principle (first stated by Pascal in 1653). It will be noticed that it applies strictly only to a fluid of constant density, that is, an incompressible fluid. If an increase of pressure affected the density to an appreciable extent, it would

cause a change in the difference of pressure between two points not at the same level. Liquids are so nearly incompressible that the principle is practically true for all liquids. Gases are more compressible, but, on account of their small density, the pressure is practically the same at all points, provided the volume be not enormously great. Hence the principle is also practically true for gases.

In the *hydraulic press* a small cylinder containing a piston is in communication with a large cylinder containing a correspondingly large piston, both cylinders being filled with a liquid. If the pressure in the liquid be p and the areas of the pistons be s and S respectively, the thrust on the small piston will be ps and that on the large piston pS. A force ps applied to the rod of the small piston will produce a force pS on the rod of the large piston. (In the forging press of the Bethlehem Steel Works a force of 14,000 tons is thus produced by a water pressure of 8000 lbs. per square inch.)

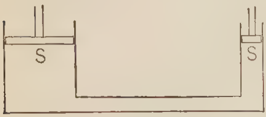

FIG. 83.

145. Thrust on a Plane Surface immersed in a Liquid. — Let the area of a plane surface immersed in a liquid be A, and suppose A divided up into a large number of parts $a_1, a_2, \cdots$ each so small that the pressure over it may be regarded as uniform. If $h_1, h_2, \cdots$ are the respective depths of these parts, and if P is the pressure on the surface of the liquid, and F the thrust on the immersed surface,

$$\begin{aligned} F &= (P + g\rho h_1)a_1 + (P + g\rho h_2)a_2 + \cdots \\ &= P(a_1 + a_2 + \cdots) + g\rho(h_1 a_1 + h_2 a_2 + \cdots) \\ &= PA + g\rho(h_1 a_1 + h_2 a_2 + \cdots). \end{aligned}$$

If the depth of the centroid of the surface (*i.e.* the centre of mass of a thin uniform disk having the shape of the surface) is H (§ 79),

$$H = \frac{h_1 a_1 + h_2 a_2 + \cdots}{A}.$$

$$\therefore F = (P + g\rho H)A.$$

Hence the thrust on a plane surface is the same as if the surface were horizontal and at the depth of its centroid.

146. Archimedes' Principle. — *The resultant force which a fluid at rest exerts on a body immersed in it equals the weight of the fluid displaced and acts vertically upward through the centre of gravity of the fluid before it was displaced.* For, call the immersed body B and the fluid displaced S. Before it was displaced S was at rest and it must, therefore, have been sustained by a force equal to its weight and acting upward through its centre of gravity. When B is introduced (the level of the liquid being kept the same), the pressure on any part of its surface is the same as the pressure that acted on the corresponding part of the surface of S. Hence B must be buoyed up by a force equal to the weight of S acting through the centre of gravity of S. The centre of gravity of the displaced fluid is called the *centre of buoyancy* of the body immersed. When the immersed body is homogeneous, its centre of buoyancy coincides with its centre of gravity.

If the volume of the body immersed is v and its density (supposed uniform) is ρ, its weight is $v\rho g$. If ρ' is the density of the fluid, the weight of the fluid displaced is $v\rho' g$ and this is, therefore, the apparent loss of weight of the immersed body. Hence the ratio of the weight of the

body to its apparent loss of weight when immersed is $\rho : \rho'$. Thus by weighing a body in air and in a liquid, the density of the body can be found if that of the liquid be known, or the density of the liquid can be found if that of the solid be known.

147. Specific Gravity and Density. — The *specific gravity* of any substance is the ratio of its density to that of water at 4° C., or the ratio of the mass or weight of any volume of the substance to that of an equal volume of water at 4° C. Since the mass of a c.c. of water at 4° C. may be taken as 1 g. (§ 53), the density of water at 4° C. is **1**, and therefore in the C. G. S. system the specific gravity of a substance is the same as its density. In the F. P. S. system the density of water is 62.4 (lbs. per cu. ft.), and the density of any substance equals its specific gravity multiplied by 62.4.

148. Hydrometers. — The common hydrometer (or hydrometer of variable immersion) is an instrument for finding the specific gravities of liquids. It is made of glass and consists of a body, in the form of two bulbs, with a tube or stem attached. The lower bulb is weighted with mercury so that the instrument will float stably with the stem vertical. When in a liquid it sinks to a depth that indicates, by a scale on the tube, the specific gravity of the liquid. To construct a suitable scale for any hydrometer the depth is noted to which it sinks (1) in

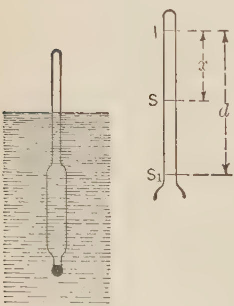

FIG. 84.

water, (2) in a liquid of known specific gravity s_1. This gives the water mark and the mark on the scale for a density s_1. Let the distance between them be d and let the distance from the water mark to the place where the s mark should be put be x. If v is the volume the instrument displaces when floating in water, the volume it displaces when floating in liquids of specific gravities s_1 and s respectively must be $\frac{v}{s_1}$ and $\frac{v}{s}$ respectively. Hence, if a is the area of cross-section of the stem and if s and s_1 are both greater than 1,

$$v - \frac{v}{s_1} = ad,$$

$$v - \frac{v}{s} = ax,$$

whence $$\frac{x}{d} = \frac{1 - \frac{1}{s}}{1 - \frac{1}{s_1}}.$$

(If s_1 and s be less than 1, the minus signs must be replaced by plus signs.)

Exercise XXXIV. Archimedes' Principle

(1) To the inner wall of a straight glass tube (a shade for a Welsbach burner) a millimetre scale on thin paper is fastened, so that the scale is parallel to the axis of the tube and the paper reaches to about 2 cm. of one end of the tube. This end is closed by a thin cork pushed a small distance into the tube and covered by a layer of paraffin wax that just fills the end of the tube.

The tube is then floated in a jar of water and disks of lead slightly smaller in diameter than the tube (or lead shot) are dropped in until the tube will float stably. The depth of the tube is read on the scale by glancing along the under surface of the water in the jar. The

tube is then removed, dried, and weighed with its contents. The volume of the tube immersed is calculated and compared with the weight. These operations should be repeated with different depths of immersion.

The same observation should be repeated with some other liquid, such as a strong brine, instead of water, and the density of the brine calculated.

(2) Find the specific gravity of aluminium (or other metal) by weighing a block of the metal, first in air and then in water, by means of a calibrated spring and a mirror scale. Then find the specific gravity of the brine used in (1) by weighing the block in the brine. Air-bubbles clinging to the block may be removed by a bent wire.

(3) Construct a scale for a hydrometer. First weight the instrument with shot so that it will float in water with the stem nearly immersed. Slip a paper millimetre scale (a strip of cross-section paper will do) into the stem (which is closed by a cork) and note the reading in water and in the brine used in (1) and (2). This gives d (§ 148). Then calculate the values of x for specific gravities increasing by .05 and lay them off on a strip of paper precisely similar to the millimetre scale. Having placed this in the instrument, test it in water and in the brine. Then make two mixtures, the first consisting of two parts (by volume) of brine and one part of water, the second with these proportions reversed. Calculate the specific gravities of these and also find them by means of the hydrometer.

DISCUSSION

(*a*) Why are the divisions of a hydrometer not equally spaced?

(*b*) The density of ice is .92. What part of the volume of an iceberg is under salt water of density 1.026?

(*c*) Archimedes weighed the crown of Hiero in water and found a decrease of $\frac{1}{14}$ of its weight, while a block of gold and a block of silver, each of the same weight in air as the crown, lost in water $\frac{4}{77}$ and $\frac{2}{21}$ respectively of the common weight. Of what did the crown consist?

(*d*) A block of metal of specific gravity 9.3 floats partly in oil of specific gravity .9 and partly in mercury of specific gravity 13.5. What part of its volume is in each?

(*e*) Explain as fully as possible the difficulty found in getting the tube in (1) to float stably when not sufficiently weighted.

149. Equilibrium of Floating Bodies. — Two forces act on a floating body: (1) the weight of the body acting at the centre of gravity G of the body; (2) the resultant upward pressure of the liquid acting at the centre of gravity C of the liquid displaced. When the body is at rest, G and C must be in the same vertical line. Let AB be the line in the body which contains G and C when the body is at rest. If the body be slightly displaced, the centre of gravity of the liquid displaced will be at some point C'. We shall consider only the case in which G, C, and C' lie in a vertical plane. The point M in which a vertical line through C' cuts AB is called the *metacentre* of the body. It is evident that if M be above G (Fig. 85), the couple acting on the body will tend to right it, and the equilibrium will be stable; but if M be below G (Fig. 86), the equilibrium will be unstable. The two cases are illustrated by a rod or a long cylinder (1) on its side, (2) on end, in water. A ship has two metacentres, — one for rolling and one for pitching motion.

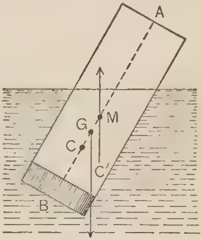

FIG. 85.

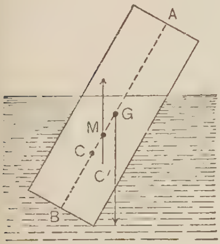

FIG. 86.

150. Flow of Liquid from an Orifice. — Liquid flows from an orifice in a vessel because the pressure in the liquid is

greater than that in the air. The escaping liquid gains kinetic energy, while the whole body of liquid in the vessel falls to a lower level and so loses potential energy. Suppose the orifice to be opened long enough for a small mass m to escape. If its velocity be v, its kinetic energy will be $\frac{1}{2}mv^2$. The state of the liquid in the vessel is the same as if the mass m had been removed from the surface and lowered to the orifice. Thus the decrease of potential energy is mgh, where h is the depth of the orifice below the surface. Hence

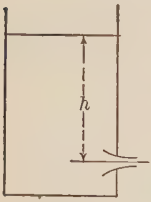

FIG. 87.

$$\frac{1}{2}mv^2 = mgh$$

or

$$v = \sqrt{2\,gh}.$$

This is also the velocity attained by a body in falling freely a distance h; this is called *Torricelli's Law*. If the escaping jet were turned vertically upward, it would rise to the level of the surface, if friction could be neglected.

The quantity of liquid that escapes in any time cannot be calculated from the value of v and the area of the orifice, owing to the fact that just outside of the orifice the jet contracts somewhat. If, however, a is the area of the smallest cross-section of the jet (called the *vena contracta*), the volume that escapes in time t is vat. The ratio of a to the area of the orifice depends on the form of the orifice and the velocity of escape, and it may be altered by the insertion of a tube (or *ajutage*) in the orifice.

Exercise XXXV. Flow of Liquid from an Orifice

A tin tank ($12'' \times 4'' \times 2''$) is mounted at the upper left-hand corner of a cross-section board as in Exercise III. The tank is filled

nearly to the brim, and an aperture in the tank is opened by pressing a lever on the side of the tank. The out-flowing liquid is caught in another tank attached to the board. The parabola of descent is obtained by slightly turning the tank, so that the water leaves a streak on the board, or by the method of Exercise III.

From the parabola the velocity of the escaping liquid is found as in Exercise III. For this calculation the values of $x^2 \div y$ obtained from points within a foot of the tank are to be preferred, since the stream breaks up, and, owing to impacts and air friction, the paths of the particles do not continue to be true parabolas. The experimental values of the velocity will in all cases be somewhat less than the velocity calculated from Torricelli's Law, and, as the difference will vary with the depth of the water in the tank, several determinations should be made. In making each determination the aperture should be open for as short a time as possible so that the level of the water in the tank may not change appreciably.

DISCUSSION

(*a*) Explain the difference between the experimental results and the values given by Torricelli's Law.

(*b*) What additional force acts on the tank when the orifice is opened?

(*c*) How high would the jet rise if it issued in some oblique direction?

(*d*) Calculate the kinetic energy of the liquid in the tank during the outflow.

(*e*) Why does the jet break up at a distance from the orifice?

(*f*) What would be the result of inserting an outflow tube to get a more definite stream?

151. Flow past an Obstruction. — When a stream meets an obstacle, the particles of the fluid are deflected from straight lines and travel past the obstacle in curves. The obstacle suffers a pressure in giving curvature to the paths of the particles just as the outer rail of a curved track suffers pressure in curving the path of a train. If the

obstacle be a vertical disk or board symmetrical about a vertical axis, the pressures on opposite sides of the vertical axis will be equal provided the disk be at right angles to the stream. If it be inclined to the stream, the "up stream" side will deflect the fluid, which will flow down along the disk, and there will therefore be a moment of force tending to set the disk across the stream. The effect may be illustrated by sweeping through the air a frame covered with paper and free to rotate about an axis, as illustrated in Fig. 88. The stability of a kite depends partly on the same principle. (Additional stability is given to a Japanese (or tailless) kite by curving the horizontal rib backward. It is readily seen from the figure that a tilt from the normal position will greatly increase the pressure on the forward side and decrease that on the other.)

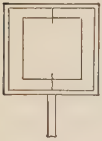
Fig. 88.


Fig. 89.

A similar effect takes place when a disk is moved through a fluid at rest. Thus a sheet of paper or a leaf, falling through air, tends to become horizontal, and the same is true of a coin sinking through water.

152. Speed and Pressure. — If the cross-section of a tube in which liquid flows steadily is not uniform, the speed of the particles of the liquid must increase as they come to a narrow part of the tube, since the same amount of liquid passes through all cross-sections. Hence there must be a resultant forward force acting on the liquid, or the pressure behind in the wider

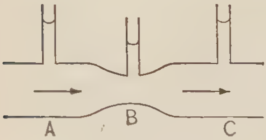

Fig. 90.

cross-section must be greater than that ahead in the contraction. Thus the pressure decreases as the speed increases, and conversely. For a similar reason, when air is forced out between two plates, the pressure between the plates is less than that outside and the plates are pressed together. (The apparatus sketched in Fig. 91, consisting of two corks and a glass tube through which air is blown, will illustrate this.) The same principle is applied in the atomizer, the steam injector, the ball nozzle, etc. From the change of pressure in a liquid as it passes through a *throat* in a tube the speed can be deduced; this is the method used in the Venturi metre for gauging flow of water.

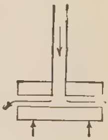

FIG. 91.

The "curve" of a rotating tennis-ball or baseball is due to the same cause. Suppose, for simplicity, that the ball is not moving forward, but is rotating about a vertical axis, and that a current of air is blowing horizontally toward it. The rotating ball is carrying a whirl of air around with it. On one side, A, the effect of the rotation of the ball is to cause a decrease in the velocity of the air blowing past the ball, while on the other side, B, it causes an increase. Hence the pressure at A is greater than that at B, and there is, therefore, a force acting on the ball in the direction AB. If now the rotating ball has a motion of translation in air otherwise at rest, the effect will be the same and the ball will have an acceleration in the direction AB and will, therefore, move in a path curved to the left in the case represented in the figure. (The "curve" of a ball may be illustrated by "serving" a toy balloon with a "cut" from the hand.)

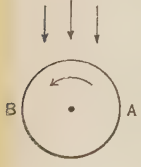

FIG. 92.

153. Viscosity. — A body of fluid continues to change in form so long as there is the smallest shearing stress acting

on it; but the rate of change of shape for a given shearing stress is different for different fluids. For example, a liquid flows down an inclined plane, however slight the inclination, but the rate of flow is different for different liquids; the shearing stress is, in this case, due to the component of gravity down the plane. Careful experiments have shown that in all cases the rate of change of shape or rate of shearing, after it has become steady, is accurately proportional to the magnitude of the shearing stress. Hence the internal frictional resistance, which just counterbalances the external force and so prevents acceleration, must also be proportional to the rate of shearing. The internal resistance is called the *viscosity* of the fluid and *the ratio of the shearing stress to the rate of shearing* is called the *coefficient of viscosity* of the fluid.

For clearness this definition of the coefficient of viscosity may be interpreted as follows: Suppose the space between two large plates A and B to be filled by a fluid. Let B be kept at rest while A is kept moving parallel to B with a steady velocity v. In a short time t, A travels a distance vt, and if the distance between the plates is d, the shear produced is $vt \div d$. Hence the rate of shearing is $v \div d$. If the area of each plate is a and the force applied to each plate is F, the measure of the shearing stress is $F \div a$. Hence, denoting the coefficient of viscosity by μ,

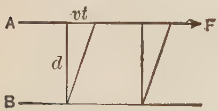

FIG. 93.

$$\mu = \frac{F}{a} \div \frac{v}{d} = \frac{Fd}{av},$$

and therefore $$F = \mu \frac{av}{d}.$$

This equation is frequently taken as the definition of μ, the other letters having the meanings already assigned to them. Taking the case in which a, v, and d are all unity, we get the following definition of μ: the coefficient of viscosity of a viscous material is *the tangential force on unit of area of either of two horizontal planes at the unit distance apart, one of which is fixed while the other moves with the unit of velocity, the space between them being filled by the viscous material* (Maxwell).

Exercise XXXVI. Viscosity

A vertical steel rod (1.5 cm. in diameter) is held between needle points, one of which is in a horizontal bar (clamped to uprights) while the other is in a plug that closes the lower end of a brass tube (about 1.7 cm. in internal diameter) containing glycerine. The tube rests on the table and its upper end is steadied by an adjustable clamp attached to a second horizontal bar. A cord that passes over a pulley and carries a scale pan is wrapped around the rod. (The pulley should be kept well oiled.)

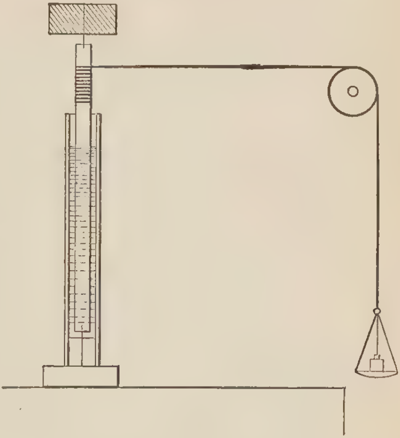

Fig. 94.

Weights are placed in the pan and the time required for the pan to descend from a definite level to the floor is observed several times with care, the pan being released exactly on a tick of the clock. The velocity of rotation of the rod becomes constant almost immediately after the release of the pan (it is just perceptible that there is a momentary acceleration which

produces a velocity too great to be maintained steadily and that the velocity falls at once to the constant value). From the distance of descent and the time the velocity is calculated. The observations should be repeated with various weights in the pan.

The various velocities should then be plotted as abscissæ with weights as ordinates. The result should be a very satisfactory straight line. This line will cut the vertical axis at a distance from the origin that represents the friction of the bearings and pulley. Subtracting this friction from all the ordinates, a line through the origin showing the constancy of the ratio of shearing stress to rate of shear will be obtained.

The friction of the bearings and pulley may be found directly by removing the glycerine and finding what weight attached to the cord will keep the axis in steady rotation (the lower bearing should be lubricated with glycerine as in the preceding).

To find the coefficient of viscosity of the glycerine the internal diameter of the tube must be known. It may be determined by weighing the tube (and plug) empty and then when filled with water. The value of a (§ 153) may be calculated from the mean of the diameter of the rod and the internal diameter of the tube. The ratio of F to v is taken from the line through the origin.

(The plug may be made so as to screw in, but this is not essential; a well-turned brass plug that can be forced in is sufficient. For the lower bearing use a large, thick-pointed needle forced into a hole drilled through the plug.)

DISCUSSION

(*a*) Meaning and definition of viscosity.

(*b*) Why does the weight not fall with an acceleration as if the resistance were ordinary friction?

(*c*) Why should the velocity become for a moment too great to be maintained?

(*d*) In what way would the motion differ if the rod carried a disk of considerable moment of inertia?

(*e*) What becomes of the potential energy of the descending weight?

(*f*) Why does a body (*e.g.* a raindrop or a parachutist) falling a long distance through the air attain a steady velocity?

154. Flow through a Capillary Tube. — When a fluid flows without eddies through a capillary tube (that is, a tube of very small bore), each particle moves in the direction of the length of the tube. All particles in a cylindrical layer move with the same velocity. Hence the flow consists in a sliding of layer over layer. There is very complete evidence that the fluid in contact with the surface of the tube does not slip on the solid, but adheres to it. Assuming this, it can be shown that, if r is the radius of the tube and l its length, and if the difference of pressure in the fluid at the ends of the tube is p, the volume that flows out of the tube in unit time is

$$V = \frac{p\pi r^4}{8\,l\mu}.$$

Numerous experiments have verified this formula for tubes of different lengths and radii and for different pressures. This shows that there is no slipping of the liquid on the surface of the tube, for such slipping would allow an outflow not included in the formula.

From the rate of outflow of a fluid through a tube of measured dimensions, the value of μ for the fluid can be deduced, and this is the most common method of measuring the coefficient of viscosity of a fluid.

SURFACE TENSION AND CAPILLARITY

155. Intermolecular Forces. — Many facts show that particles of any form of matter attract one another with very great forces when they are very close together, but these forces decrease so rapidly with distance that they become negligible beyond a certain distance called the *range of*

molecular forces. Fracture of a brittle body consists in slightly separating the particles of the body so that they cease to attract. Two metallic surfaces may be brought quite close together without showing any attraction, but if brought very close by pressure or welding, they will adhere. A sheet of glass brought toward a water surface is not attracted sensibly until it touches the water, when the latter adheres to it. If water, carefully freed from air, be heated so as to fill a tube with a very fine stem which is then sealed off, the water will sometimes continue to cling to and fill the tube in spite of its tendency to contract as it cools.

The range of molecular forces is not yet well ascertained, but there is reason to believe that it is about .00005 mm. A sphere of this radius is called the *sphere of influence* of the particles.

156. Surface Layer of a Liquid. — A particle p in the liquid at a greater distance from the surface than the range of molecular forces r is equally attracted in all directions, and the resultant force of attraction on it is zero. But a particle p_1 at a distance less than r from the surface is more attracted inward than outward. For, with p_1 as centre, describe a sphere with r as radius. That part, abc, of the sphere which is out of the liquid contains only a relatively very small number of particles of air or vapor. Let ed be a plane through p_1 parallel to the surface, and let fg be another plane parallel to the surface

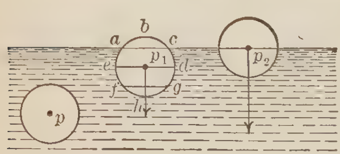

FIG. 95.

and cutting off a segment *fgh* equal to *abc*. The attractions on p_1 of the particles in *acde* and *defg* will neutralize one another, and the attractions of the particles in *fgh* will constitute an unbalanced force inward acting on the particle p_1. A particle, such as p_2, in the surface will be attracted inward by the resultant of the attractions of all the particles in a hemisphere of the sphere of influence.

157. Tendency of Surface to Contract. — The effect of the inward attractions on the particles near the surface of a liquid is a tendency of the surface to contract to the form of smallest area possible under the circumstances. For a given volume the form of smallest surface is a sphere, and this is accordingly the form that a body of liquid assumes when other forces, such as gravity, do not interfere. For example, a small drop of mercury on a glass plate is practically spherical; its weight and the pressure of the plate are too slight to cause any appreciable flattening. A drop of any liquid descending slowly through a liquid with which it does not mix is spherical, and the same is practically true of a falling raindrop. A soap-bubble consists of a thin film of liquid with two spherical surfaces. Lead shot, solidifying from the liquid form while falling through air, are spherical. The round form of the melted end of a stick of sealing-wax is due to an attempt to assume the spherical form. The hairs of a camel's-hair brush stand apart when the brush is plunged in water, but as soon as the brush is drawn out the film of water on it draws the hairs together.

158. Surface Tension. — Many examples of this tendency of the surface to contract show the existence of a contrac-

tile force, the direction of which is parallel to the surface. A loop of silk placed on a film of a soap solution is drawn out into a circle as soon as the part of the film inside of the loop is ruptured. The unbroken part of the film shrinks to a minimum, and so causes the loop of thread to enclose the largest area possible for a loop of given length. To accomplish this, it must pull on the loop with a force parallel to the surface of the film.

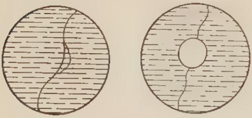

Fig. 96.

As another example, consider a film of a soap solution on a rectangle of wire *abcd*, one side of which, *cd*, is free to slide parallel to itself. To keep *cd* at rest, a force F away from *ab* must be applied to it. Hence there must be a tension in the film tending to draw *ab* and *cd* together. This tension exists only in the two surfaces, for the force F is found to be the same whether the film be a thick one or a thin one. (This statement is not quite exact in the case of the thinnest films possible, for in this case the thickness of the whole film is less than the range of molecular forces, and this produces complications which cannot be considered here.) Thus the tension in a liquid film does not increase when the film is stretched, whereas the tension of an elastic membrane is increased by stretching.

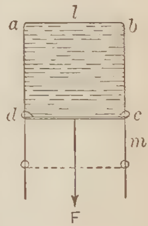

Fig. 97.

The tension or contractile force across each unit of length on the surface of a liquid is called the *surface tension* T of the liquid. The surface tension of pure distilled water at 0° C is 75 dynes per centimetre, of ether 19 dynes per centimetre.

159. Surface Energy. — When the surface of a quantity of liquid is increased, more particles are brought out into the surface layer, and, since the inward attraction has to be overcome in doing this, work is necessary to increase the surface area of a liquid. Thus an increase of surface must result in an increase of energy in the liquid, the increase of energy being proportional to the increase of surface, and therefore called *surface energy*. Let the amount of this energy per unit surface be E. For an increase s in the surface the increase of energy will be Es.

Consider again the last example of § 158. Suppose that by the application of the force F or $2\,Tl$ the wire cd is moved a distance m away from ab. The work necessary to do this is $2\,Tlm$. Now lm is the increase in each surface, and $2\,lm$ is the whole increase of both surfaces, say s. Hence the increase of energy is Ts, and this must be equal to Es. Hence $T = E$, or the surface tension is equal to the surface energy per unit area. Surface tension can be calculated from the force required to stretch a film, and in various other ways (§ 167). Each such measurement is also a measurement of surface energy.

It should be noticed that the reasoning of the first paragraph of this section applies also to solids, so that in a solid also there must be a certain amount of energy located in the surface and proportional to the surface. A solid probably also has a surface tension, that is, a force along the surface that has to be overcome in increasing the surface, *e.g.* in stretching a wire, but it cannot be measured owing to the fact that the force required to stretch the internal parts of the solid is incomparably greater, and the two forces cannot in measurement be separated.

160. Angle of Contact. — Where the surface of a liquid meets that of a solid it forms with it a definite angle called the *angle of contact.* The angle of contact of clean water and clean glass (Fig. 98) is 0°, that is, the curved water surface is tangential to the glass. If the surfaces are not clean, the angle may be large. For clean mercury and

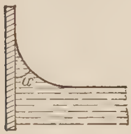

Fig. 98.

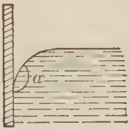

Fig. 99.

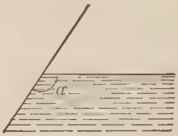

Fig. 100.

clean glass (Fig. 99) the angle of contact is about 145°, but varies considerably with slight contamination.

The angle of contact α can be measured in various ways. The simplest (but not the most accurate) is to tilt the surface of the solid until the surface of the liquid is horizontal (Fig. 100) right up to the solid surface and then measure the angle of tilt. From this α can be readily deduced.

Our knowledge of the forces between molecules is so imperfect that we cannot yet give a full explanation of the curvature of the surface of a liquid in contact with a solid; but the existence of surface energy (§ 159) affords some help. Surface energy is a form of potential energy, and bodies free to move are not in stable equilibrium unless their potential energy is a minimum (§ 105). Hence a liquid in contact with a solid will show a tendency to spread over the latter, as in Fig. 98, if the energy of the surface of the solid is less when the surface is covered by the liquid than when it is not covered. When the opposite is the case the liquid will tend to recede and leave the solid uncovered as in Fig. 99. The extent to which the liquid curves in either case is limited by the fact that curvature increases the free

surface of the liquid and so produces an increase of the total energy in that surface.

161. Pressure on a Curved Surface of a Liquid. — An elastic band stretched around a cylinder presses on the cylinder, and the cylinder presses back on the band with an equal force. To support the band the pressure on its concave side must exceed that on its convex side. A curved liquid surface is also in a state of tension, and for equilibrium the pressure on its concave side must exceed that on its convex side. The difference of pressure on the two sides of a curved liquid surface can be stated in terms of the curvature of the surface and its surface tension.

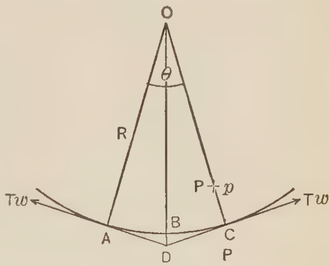

Fig. 101.

For simplicity we shall consider first the case of a surface curved like the surface of a circular cylinder (this is the shape of the surface of a liquid between two parallel plates standing close together in the liquid — see Fig. 102). Let a plane perpendicular to the length of the cylinder cut the liquid surface in ABC and the axis of the cylinder in O. Consider a short and very narrow strip of the surface in the form of a curved rectangle of which ABC is one edge and denote the small angle AOC by θ. The thrust on each minute part of the strip is along the radius. Since θ is small these thrusts are practically parallel and their resultant equals their sum and acts along the bisecter of θ.

If R is the radius of the cylinder, the length of the

strip (in the direction of ABC) is Rθ, and if its width is w, its area is $R\theta w$. If the pressure on the concave side exceeds that on the convex side by p (per unit area), the resultant thrust on the strip is $pR\theta w$. The surface tension T exerts a force Tw on each end of the strip. For equilibrium the lines of these three forces must intersect in a point D (§ 95), and the sum of the components of the forces Tw along DO must be equal and opposite to the resultant thrust. The component of each force Tw along DO is $Tw \cos \frac{1}{2} ADC$, or $Tw \sin \frac{1}{2}\theta$. But since θ is very small, we may put $\frac{1}{2}\theta$ for $\sin \frac{1}{2}\theta$. Hence

$$pR\theta w = 2\,Tw\,\tfrac{1}{2}\,\theta$$

or

$$p = \frac{T}{R}.$$

162. Level of Liquid between Two Plates. — Liquid in contact with a plate meets the latter at a definite angle, and is, therefore, curved upward or downward. The same is true of a second plate close to the first, and if the plates are close together, the two curvatures join together to form a cylindrical surface of radius R. If the surface is concave upward, the pressure on the upper side is atmospheric pressure P, and the pressure at a point C just beneath the curved surface is $P - \dfrac{T}{R}$. The pressure at a point A in the free surface outside of the plates is P, and the pressure at a point B between the plates and at the same level as A must also be P. Hence the

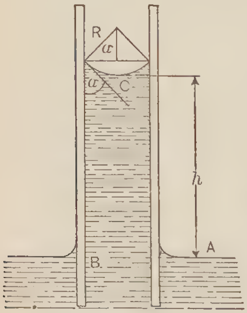

Fig. 102.

pressure at C is less than that at B, and C must, therefore, be at a higher level than B. Let the difference of level of C and B be h. Then the pressure at B equals that at C plus $g\rho h$, ρ being the density of the liquid. Hence

$$P - \frac{T}{R} + g\rho h = P.$$

$$\therefore\ h = \frac{T}{g\rho R}.$$

A radius of the cylinder through a point in the line of contact makes with a line perpendicular to the two plates an angle equal to the angle of contact α. Hence, if d is the distance between the plates, $R \cos \alpha = \frac{1}{2} d$, and, therefore,

$$h = \frac{2\, T \cos \alpha}{g\rho d}.$$

The same formula is obtained if a case in which a liquid is depressed between the plates is considered. For such a liquid, $\alpha > 90°$ and $\cos \alpha$ is negative.

It has been supposed in the above that h is measured to the bottom of the curved surface between the plates. Some of the liquid is actually at a higher level, and it can be shown that for greater accuracy h should be increased by about $\frac{1}{10} d$ if $\alpha = 0$. This correction is, however, usually negligible.

163. Liquid between Two Inclined Plates. — If two vertical rectangular plates, standing in a liquid, touch along a vertical line and are inclined to one another at a small angle θ, the separation of the plates at any distance x from the line of contact is $x\theta$, and the plates are so nearly parallel that we may apply the formula of § 162

to find the rise of liquid between them. Denoting the elevation at a distance x by y,

$$y = \frac{2\,T\cos\alpha}{g\rho x\theta}.$$

$$\therefore\ xy = \frac{2\,T\cos\alpha}{g\rho\theta}.$$

The right-hand side of this equation is a constant for a given inclination of the plates. Hence xy is constant. This is the characteristic of a rectangular hyperbola. Hence this is the curve formed by the surface of the liquid.

Exercise XXXVII. Surface Tension

Two sheets of plate glass are prepared for a study of the rise of water between them, as explained in § 163. One should be somewhat larger than the other, and a tank should be formed by cementing strips of glass to the bottom of the larger. (Strips of adhesive paper such as are used for the edges of lantern slides may be used to bind the edges together. These strips should be then covered by a thin film of paraffin wax, applied melted, to prevent wetting by the water, and the tank should be made water-tight by coating the inside edges with paraffin wax.) The edge of the smaller glass plate that is to be in contact with the other plate should be ground very smooth and straight (by rubbing it on a sheet of sandpaper, tacked to a board and wet with turpentine).

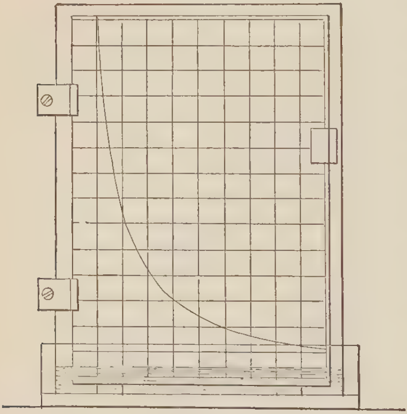

Fig. 103.

Fasten a sheet of cross-section paper to the vertical board, taking care that the lines are truly vertical and horizontal. In front of this mount the larger glass plate, allowing it to rest on a small platform clamped to the board. Wash both plates and the inside of the tank clean with chromic acid and distilled water. Then stand the smaller plate in the tank and level the supporting platform until the line in which the plates meet is vertical. The spring clips shown in the figure are for the purpose of keeping an edge of the small plate tight against the large plate. A small strip of steel of definite thickness (about 1 mm.) is used to separate the other edge of the plates.

After filling the tank nearly full with distilled water, allow the plates to come closer together than they are intended to remain, and then separate them cautiously until the strip of steel can be inserted. Push the latter in some distance, and the result will be a clear, smooth curve, formed by the surface of the water between the plates. Find the abscissæ and ordinates of various points on the curve, and calculate the values of xy. The value of θ is found from the thickness of the steel strip and its distance from the edge of contact of the plates. The observations should be made as quickly as possible in order that α (which is 0 while the plates are well wet) should not markedly change. T should be calculated from the mean value of xy. (Cross-section lines etched on the front of the larger plate are better than the cross-section paper, but are not indispensable.)

DISCUSSION

(*a*) Why does the liquid rise between the plates?

(*b*) Explain the force that urges the plates together.

(*c*) Calculate the pressure where $x = 5$ cm. and $y = 2$ cm.

(*d*) Sources of error in the value of T.

(*e*) Is the value found for T more probably too high or too low?

(*f*) How high would the water rise if the smaller plate were placed with its lower horizontal edge in contact with the other plate? (This might be tried and the value of T deduced.)

(*g*) How could the quantity of liquid that rises between the plates be calculated?

164. Pressure of Curved Liquid Surfaces. — A cylindrical surface of radius R and tension T exerts a pressure $\frac{T}{R}$ on the concave side. Such a surface is produced by bending a plane surface once, hence it is called a surface of *single curvature*. If to a small part of a cylindrical surface a second curvature be given by bending it in a direction at right angles to the first direction of bending, it will become part of a surface of *double curvature*, such as a sphere, spheroid, etc. (It will also be curved in intermediate directions, but this is merely a consequence of the two *principal* curvatures.) If the radius of the second curvature be R', the tension will cause a second pressure $\frac{T}{R'}$, and the whole pressure will be

Fig. 104.

$$p = T\left(\frac{1}{R} + \frac{1}{R'}\right).$$

In the case of a sphere $R = R'$. Hence $p = \frac{2\,T}{R}$. For a spheroid ellipsoid, etc., R and R' are unequal (except at certain particular points). There are surfaces, such as a saddle-back or spindle, at any point on which the two curvatures are in opposite directions, and R and R' therefore differ in sign.

A spherical soap-bubble consists of a thin sheet of liquid between two contractile spherical surfaces of practically equal radii R and under a tension T. Hence the pressure inside must exceed that outside by $\frac{4\,T}{R}$. If a soap-bubble be formed between two glass funnels so that it is not spherical, the excess of internal pressure must be $2\,T\left(\frac{1}{R} + \frac{1}{R'}\right)$. If the small ends of the funnels be open to the atmosphere, there can be no excess of internal pressure, and therefore $\frac{1}{R} + \frac{1}{R'} = 0$, or $R = -R'$. This accounts for the spindle shape of such a film.

165. Level of Liquid in a Capillary Tube. — The surface of liquid in a vertical capillary tube (that is, a tube of very

small bore) of circular cross-section is spherical. Hence the pressure on the concave side must exceed that on the convex side by $\frac{2\,T}{R}$, where R is the radius of the spherical surface. Therefore the liquid will rise above the ordinary level if the concavity is upward, and it will be depressed if the concavity is downward. The elevation or depression is found exactly as in § 162, in fact, it is only necessary to substitute $\frac{2\,T}{R}$ for $\frac{T}{R}$. Hence

$$h = \frac{2\,T}{g\rho R}.$$

If the angle of contact is α and if the radius of the tube is r, then $R \cos \alpha = r$. Hence

$$h = \frac{2\,T \cos \alpha}{g\rho r}.$$

166. Other Effects of Surface Tension. — Two bodies, floating close together on a liquid that wets both, apparently attract one another. The liquid rises between them, and while the pressure in the elevated liquid between them is less than atmospheric pressure P, the pressure at the same level on their other faces is P. Hence

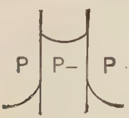

Fig. 105.

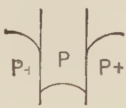

Fig. 106.

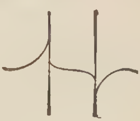

Fig. 107.

they are urged together. If neither body is wet by the liquid, the liquid is depressed between them. Thus for a certain space between them the pressure is P, while at the same depth on their other faces the pressure is due to liquid below the free surface, and therefore exceeds P. Hence they are pushed together. If one is wet by the liquid while the other is not, the form of the liquid surface is as shown in Fig. 107, and it is readily seen that each plate is urged away from the other.

A piece of camphor dropped on clean water begins to dissolve. At some points the solution proceeds faster than at other points. At places where the water is most polluted by the camphor the surface tension is most weakened, and thus the camphor is drawn away by the stronger tension on the opposite side. Hence rapid and erratic motions of the camphor ensue. A very slight film of oil in the water weakens the surface tension so much that such motions do not take place.

167. Methods of measuring Surface Tension. — The most common method is to observe the rise of the liquid in a capillary tube and use the equation of § 165. Other methods that have been used depend on the downward pull of the liquid on a thin plate partly immersed in the liquid, or on the pull required to draw a plate away from the liquid, or on the size of drops falling from a tube, or on the form of a drop (of mercury) resting on a glass plate. Probably the most accurate method is by observation of the wave length and frequency of small waves or ripples on the surface of the liquid, for the propagation of such waves depends on surface tension.

168. Diffusion; Osmosis. — When two liquids that can mix are placed in contact, the particles of each begin to pass into the other. This process is called *diffusion*. If a vial filled with a solution of some salt (*e.g.* blue solution of copper sulphate) be wholly immersed beneath water in a beaker, the salt will slowly diffuse out of the vial. The quantity of salt that leaves the vial depends on the time, the strength of the solution, and the temperature, and it is also markedly different for different salts or dissolved substances. The rate of diffusion of one salt is practically independent of the presence of other salts, provided there is no chemical action.

Substances can be roughly classified according to their rates of diffusion into *crystalloids*, such as mineral salts and acids, which diffuse rapidly, and *colloids*, such as starch, albumen, and caramel, which diffuse slowly. The difference is probably due to the fact that the molecules of colloids are larger and, therefore, move more slowly than those of crystalloids. Colloids in water tend to form jellies, which apparently consist of a more or less solid framework, through which the liquid is dispersed. Through such a jelly, or colloid membrane, crystalloids can diffuse, while colloids cannot. Wet parchment or bladder is a colloid, and a mixture of crystalloids and colloids in water (*e.g.* the contents of the stomach in cases of poisoning) can be separated by placing the mixture in a tube closed below by parchment and dipping it in a vessel of water. When a colloid membrane separates water and an aqueous solution, the pure water passes more readily than the water of solution. If, for example, a tube closed below by parchment be partly filled with a sugar solution, and be dipped in water so that both liquids are at the same level, the level will continue for some time to rise in the tube; water particles pass through the membrane in both directions, but more pass into the tube than in the reverse direction.

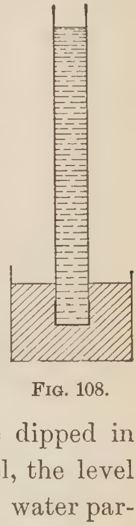

FIG. 108.

Certain membranes, called *semipermeable* membranes, allow water, but not dissolved salts, to pass. A layer of ferrocyanide of copper deposited chemically in the pores of a porous earthenware plug that closes the lower end of a tube is an example. If the tube be filled by a salt solu-

tion and be immersed in water, the water will continue to enter the tube and will rise until the pressure of the column in the tube prevents further inflow. The height in the tube depends on the particular salt, the strength of the solution, and the temperature; the pressure of the column ($g\rho h$) when it ceases to rise is called the *osmotic pressure* of the dissolved salt. The explanation of the action is not yet certain, but one interesting law has been arrived at, namely, that the osmotic pressure of a salt in a very weak solution is equal to the pressure which the particles would exert if the water were supposed absent and the particles were in the gaseous state.

GASES

169. Gases. — The shear modulus of a gas, like that of a liquid, is zero. Hence all the properties of fluids that depend on the absence of elasticity of form are possessed by gases. Thus the pressure of a gas on any surface is perpendicular to the surface (§ 139), the pressure at any point is the same in all directions (§ 141), and an increase of pressure at any point in a gas at rest is accompanied by an equal increase at all points (§ 144). Archimedes' principle (§ 146) is also true of gases. A balloon is sustained by a force equal to the weight of the air which it displaces. When the weight of a body is to be found with great accuracy, allowance must be made for the weight of the air displaced by the body and also for the weight of the air displaced by the weights used. The pressure in a gas also increases with the depth, and the law of increase is the same as in the case of a liquid (§ 142). All gases are viscous, and the definition of the coefficient of

viscosity of a gas is exactly the same as that of a liquid (§ 153).

170. Pressure of the Atmosphere. — An important case of pressure due to gravity and depth is the pressure of the atmosphere. If a very long tube were supposed to extend from the surface of the earth to the outer limit of the atmosphere (200 miles or more), the pressure at the bottom of the tube would equal the weight of the air in the tube. We cannot calculate the pressure directly by means of the formula $p = g\rho h$, since the density is different at different heights. We can, however, find the pressure directly by balancing it against the pressure produced by a column of some dense fluid such as mercury.

Fig. 109.

A pressure-gauge for measuring the pressure of the atmosphere is called a *barometer*. One form (Bunsen's) consists of a U-tube having a long closed arm occupied only by mercury and a shorter arm partly occupied by mercury and open to the atmosphere. If the long arm is of sufficient length, there will be a vacuum above the mercury, and the pressure at the level of the surface will consequently be zero. If the difference of the level of the surfaces in the two arms is h and the density of mercury is ρ, the pressure in the mercury surface in the short arm is $g\rho h$, and this for equilibrium must also be the pressure of the atmosphere.

Another form of barometer, called the *cistern barometer*, consists of a straight tube filled with mercury. The press-

ure at the surface of the pool is atmospheric pressure and the equal pressure at the same level in the tube is that due to a column of mercury equal to the difference of level of the two mercury surfaces, or $g\rho h$. When the atmospheric pressure increases, the level of the mercury in the tube rises and that in the cistern falls. By reading on a scale etched on the glass tube or on a separate scale placed beside the tube, h may be found, but this will require a reading of each of the two mercury surfaces. In Fortin's barometer this double reading is avoided by using a cistern with a flexible leather bottom; by adjusting a screw that presses on the leather the level of the mercury in the cistern may be brought to the zero of the scale.

The *aneroid* barometer is a shallow, cylindrical, metal box exhausted of air; the top rises and falls with changes of atmospheric pressure, and its motion is communicated, by a magnifying system of levers, to an index that indicates the pressure on a scale which is graduated by comparison with a mercury barometer.

171. Corrections of Barometer Reading. — For many purposes it is necessary to compare the atmospheric pressure at different times and at different places. To do this it is not sufficient merely to compare the heights of the barometer, for, since $P = g\rho H$, variations in the values of ρ and g should be allowed for. Moreover, H is measured on a scale, the length of each unit of which depends on the temperature. To allow for these differences it is customary to calculate from H what the height, say H_0, of the barometer would have been, if (the actual pressure remaining the same) the temperature of the mercury and the scale had been that of melting ice, or zero on the centigrade scale, and g had been equal to the acceleration of gravity at sea-level in a latitude of 45°, say g_0. Let ρ be the density of mercury at the actual temperature $t°$ C. and ρ_0 its density at 0° C.; also let each unit

of the scale equal n true units of length (centimetres or inches) at t° C., and n_0 true units at 0° C. Since the pressure may be expressed either as $g\rho Hn$ or as $g_0\rho_0 H_0 n_0$,

$$H_0 = \frac{g}{g_0} \cdot \frac{\rho}{\rho_0} \cdot \frac{n}{n_0} H.$$

Now if λ is the latitude of the place of observation and l its height above sea-level in metres (§ 56),

$$\frac{g}{g_0} = 1 - .0026 \cos 2\,\lambda - .0000003\,l.$$

It is shown in works on Heat that (assuming the scale to be of brass)

$$\frac{\rho}{\rho_0} = (1 - .000181\,t) \text{ and } \frac{n}{n_0} = (1 + .000019\,t).$$

Multiplying these factors together and neglecting the products of small quantities, we get

$$H_0 = (1 - .000162\,t)(1 - .0026 \cos 2\,\lambda - .000000\,3\,l) H.$$

This value of H_0 is in scale units at 0° C. If the scale is correct at 0° C., no further correction is required; if it is not, H_0 must be multiplied by the ratio of the scale unit at 0° C. to the centimetre or inch.

A unit sometimes employed in stating pressures is the *standard atmosphere*, that is, the pressure of a column of mercury 76 cm. high, at 0° C. at sea-level in the latitude of 45°.

Another source of error in estimating the pressure by a cistern barometer is curvature of the surface of mercury in the tube. The curvature being downward, the surface tension causes a downward pressure on the mercury column, thus causing it to be somewhat depressed. To get the true barometric height, the observed height must be corrected by adding the amount of this depression. The magnitude of this correction is negligible for tubes of 2.5 cm. or more in diameter. A table giving the amount of the depression for tubes of various sizes has been drawn up from comparisons of various barometers with a barometer the tube of which is so large that the depression is negligible. (See the *Smithsonian Tables*, p. 124.)

172. Pumps. — In the common lift pump water is raised by atmospheric pressure. A piston P moves in a cylinder C, which is connected by a pipe Q to the water in the well. A valve V_1 at the bottom of the cylinder and a valve V_2 in the piston open upward. When P is raised, V_1 opens and V_2 closes. Air from Q passes into C, and the pressure in Q being diminished, the water rises in Q. When the piston is forced down, V_1 closes and air is forced out through V_2. After a few strokes, water rises into C, and when P descends the water passes into the part of the cylinder above P. Thereafter at each stroke water flows out through the spout R. If the length of the tube Q be too great, water will not rise in the cylinder. Since mercury will rise to about 76 cm. in a vacuum, and mercury is 13.6 times as dense as water, water will rise in a vacuum to a height of about 76×13.6 cm. or 1034 cm., or 33.9 ft. As a matter of fact, the suction pump fails at a height less than this; for, even if there is no leakage between the piston and the cylinder, the water itself contains some air in solution, and the air separating out causes a pressure above the column of water.

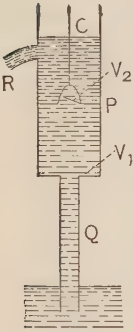

FIG. 110.

By means of the *force pump* water may be raised to a great height. In this pump there is no valve in the piston, and water is forced up a side tube R as the piston descends. A valve in R prevents the

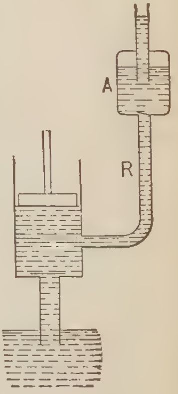

FIG. 111.

return of the water to C as the piston rises. The outflow through R takes place only during the downward motion of P, but if an "air chamber" A be inserted in R, the air, being constantly under pressure, will cause a continuous outflow.

173. The Siphon. — If the ends of a U-shaped tube full of liquid be closed and the tube be then inverted, and one end be immersed in liquid, the liquid will flow out when the ends are opened, provided the end in air be at a lower level than the surface of the liquid in the vessel. Let the depth of the open end A below the surface of the liquid in the vessel be h. Before A was opened the pressure in the liquid at A was greater than atmospheric pressure by $g\rho h$, and when A was opened the opposing pressure was only atmospheric pressure. The siphon can be used on a large scale for drainage, provided no part of the tube need be at a greater distance above the level of the liquid than the height to which water will rise in a vacuum.

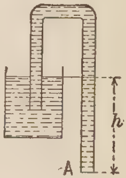

FIG. 112.

174. Boyle's Law. — Let p be the pressure and v the volume of a mass of gas the temperature of which is kept constant. *The product* $pv =$ *a constant*, no matter how much p and v may separately change. This law, discovered by Robert Boyle in 1662 and verified by him both for pressures greater than atmospheric pressure and for pressures less, is called *Boyle's Law.* (The apparatus used by Boyle was not essentially different from that used in the next exercise.)

It is evident that Boyle's Law may also be stated as follows: *the pressure of a gas at constant temperature varies inversely as its volume.* For a greater mass at the

same pressure the volume will be proportionally greater. Hence

$$pv = km,$$

m being the mass and k a constant. Now $m \div v$ is the density ρ of the gas. Hence, for a constant mass of gas,

$$p = k\rho.$$

Exercise XXXVIII. Boyle's Law

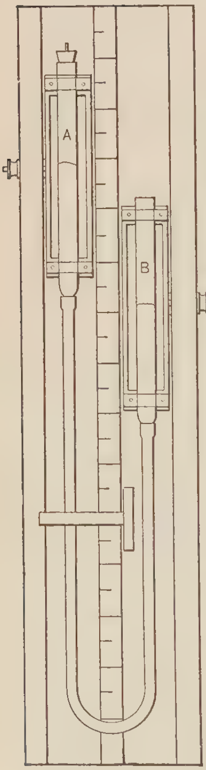

Fig. 113.

Two glass tubes A and B (Fig. 113) are mounted on blocks that can be clamped at various positions along a vertical scale. Between each tube and block is a strip of mirror glass. A rubber tube connects the lower ends of the glass tubes. In the upper end of A is a perforated rubber stopper which is shellacked before being inserted. The perforation in the stopper can be closed by a round nail coated with a mixture of beeswax and vaseline to make it air-tight. The frame of the apparatus can be levelled until the scale is vertical as indicated by a plumb-line.

The glass tubes are first brought to about the same level and mercury is poured in until it about half fills each tube. A drying tube (containing calcium chloride) is then inserted into the perforation in A, and A is filled with dry air by alternately raising and lowering B several times. The nail is then inserted in A so that the lower end just appears below the cork.

The pressure in A is less or greater than atmospheric pressure according as A is higher or lower than B. The level of the mercury in A and in B is read on the scale by means of a small T-square, one arm of which is pressed

against the framework at such a level that an edge of the other (horizontal) arm, its reflection in the mirror, and the surface of the mercury seem to coincide; the position of the horizontal arm on the scale (which it touches) should be read with the greatest care. The level of the lower end of the stopper in A is found in the same way. From these readings the pressure in A and the length of the column of air in A (which is proportional to the volume of the air) are deduced. These readings should be made with A at the top of the scale and B at the bottom, and then with A and B in various intermediate positions, and finally with A at the bottom of the scale and B at the top. From the readings thus made the constancy of pv can be tested.

DISCUSSION

(*a*) Sources of error.

(*b*) Do the tubes need to be of the same diameter?

(*c*) Why should A not be of very small bore?

(*d*) Does the stretching of the rubber tube affect the result?

(*e*) Why should the air be dry? Does it need to be perfectly dry? What would be the effect of a film of water on the mercury in A?

(*f*) How could the volume (and density) of a quantity of a powder (gunpowder, sugar, salt) be found by placing it in a vessel suspended in A?

175. Deviations from Boyle's Law. — For many gases, such as oxygen, hydrogen, and nitrogen, Boyle's Law is so nearly exact that for most purposes it may be taken as perfectly accurate. Careful study has, however, shown that it is in no case perfectly exact. The general nature of the deviations is shown by Fig. 114. The continuous line shows the connection between pressure and volume as found by careful experiment, while the dotted line indicates what

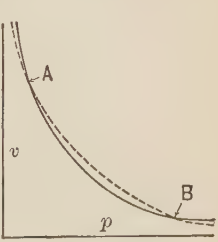

FIG. 114.

the connection would be if Boyle's Law were accurately followed. These two curves intersect in two points A and B. Consider the case of air, and suppose that when the mass of air is in the condition represented by the point A, its pressure is one atmosphere. When it is brought to the condition represented by B, the product of its pressure and volume is the same as if it had perfectly followed Boyle's Law, its pressure being 152 atmospheres and its volume $\frac{1}{152}$ of its volume at one atmosphere. The smallest value of pv between these limits occurs at about 78 atmospheres and is .98 of the product pv at A and B. Other gases show similar deviations, but the details are different for different gases.

Very careful experiments have shown that the relation between the pressure and volume of a gas is more accurately represented by the formula (due to van der Waals),

$$\left(p+\frac{a}{v^2}\right)(v-b) = \text{a constant},$$

a and b being small numbers the magnitudes of which are different for different gases.

176. Modulus of Elasticity of a Gas. — While the shear modulus of a gas is zero (§ 138), the bulk modulus has a definite value, and it is accordingly the latter that is always meant when the modulus of elasticity of a gas is referred to. Let the pressure and volume of a mass of gas be p and v, and suppose that a small increase of pressure α produces a decrease β in the volume. Then, by Boyle's Law,

$$(p+\alpha)(v-\beta) = pv.$$

Neglecting the product $\alpha\beta$ of the two small quantities α and β, the above reduces to

$$v\alpha - p\beta = 0.$$

Now the bulk modulus is the ratio of the increase of pressure to the fractional decrease of volume, that is, the ratio of α to $\frac{\beta}{v}$, and from the above this is equal to p. Hence the modulus of elasticity is numerically equal to the pressure. It should be noticed that the temperature is supposed to be constant.

177. Kinetic Theory of Gases. — The properties of gases are consistent with the view that a gas consists of particles moving with great velocities in the space occupied by the gas, that the impacts of these particles on the walls of the containing vessel produce the pressure of the gas, and that the coefficient of restitution at each impact is 1 (§ 112). The evidence for this theory is the fact that it will explain the chief properties of gases, and as an illustration we shall show that it leads to Boyle's Law.

Let a rectangular vessel of edges a, b, and c contain a single gas, and let the mass of each particle be m, and let the number of particles in unit volume be n, so that if N is the whole number of particles in the vessel, $N = nabc$. For brevity denote the two faces perpendicular to a by A and A'. Resolve the velocity V of any particle into components u, v, and w in the directions of a, b, and c, respectively. The pressure on A depends only on the u-components of the velocities of the particles. After striking A with

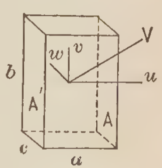

FIG. 115.

a velocity u, a particle rebounds with a velocity $-u$, and its change of velocity is $2u$. Hence its change of momentum is $2mu$, and this is therefore the momentum received by A. The particle traverses the distance a between A and A' in time $\frac{a}{u}$, and after rebounding from A' it again strikes A at a time $\frac{2a}{u}$ after the first impact. Hence it impinges on A $\frac{u}{2a}$ times per second, and the total momentum it imparts to A in a second is, therefore, $\frac{mu^2}{a}$. For any other particle u is different, but m and a are the same. Hence the total momentum imparted to A in a second, that is, the whole force on A, is $\frac{m}{a}\Sigma u^2$. Dividing this by the area bc of A we get for the pressure p on A the expression $\frac{m}{abc}\Sigma u^2$. Now $N = n \cdot abc$. Hence $p = mn\frac{\Sigma u^2}{N} = mn\overline{u^2}$, where $\overline{u^2}$ is the mean value of u^2 for all the particles. But $V^2 = u^2 + v^2 + w^2$, and, since the number of particles is very large and they are moving at random, the mean values of u^2, v^2, and w^2 are equal. Hence, denoting the mean value of V^2 by $\overline{V^2}$, $\overline{V^2} = 3\,\overline{u^2}$.

Hence
$$p = \tfrac{1}{3} mn\overline{V^2}$$
$$= \tfrac{1}{3}\rho\overline{V^2},$$

since mn is the whole mass in unit volume, that is, the density ρ of the gas. Now there is reason to believe that $\overline{V^2}$ is constant, provided the temperature does not change. Hence we see that at constant temperature the pressure of a gas is proportional to its density, which is Boyle's Law.

Since p and ρ can be measured experimentally, $\overline{V^2}$ can be deduced. From this the mean velocity (or more accurately the square root of the mean squared-velocity) can be deduced. For hydrogen at 0° C. it is 1843 metres per second, and for carbonic acid 392 metres per second.

If several different gases be present in the same enclosure, each will exercise its own separate pressure, and the total pressure p will be the sum of the separate or partial pressures p_1, p_2, $\cdots$ of the different gases. The statement, which like Boyle's Law is not perfectly exact, is called *Dalton's Law.*

178. Air-pumps. — For removing gas from a vessel, pumps are used, which are identical in principle with the suction pump used for water (Fig. 110). The efficiency of such pumps is limited by a variety of defects. Some gas leaks in between the piston and the cylinder, and the piston cannot be brought to such close contact with the bottom of the cylinder as to expel all the gas between them. For these and other reasons, when a very high degree of exhaustion is required, pumps on a different principle are used.

FIG. 116.

In the Geissler-Toepler pump, mercury in a glass tube is used instead of the piston and cylinder of the mechanical pump. Fig. 116 shows its general principle. The tube C is connected with the vessel to be exhausted. The long flexible rubber tube G connects the mercury reservoir H to the glass bulb A. When

H is raised, mercury enters A, B, and C, and seals the connection between A and C. As H is further raised, the gas in A is driven out through D, and escapes from beneath the mercury in the vessel E. When H is lowered, the mercury rises in E and prevents the entrance of air, and gas is drawn in through C and fills the bulb A. When H is raised again, the gas in A is expelled through D, and so the process is continued. When nearly all the gas in the vessel connected with C has been removed, there will be very little pressure in A and the tube connected with it. Hence the mercury will rise to nearly barometric height in D, which must be 76 cm. or more in length. To completely expel the gas from A, B, and D, H must be raised to such a height that the mercury will pass over into D; and to prevent mercury passing out through C, the latter must be very long. The purpose of B is to prevent danger of A being broken by the sudden inrush of gas through C as H is lowered.

Mercury pumps are used to exhaust incandescent lamp bulbs. By such pumps it is possible to reduce the pressure in a vessel to .00001 mm. of mercury. For measuring such low pressure a special gauge (the Macleod gauge) is used.

179. Effusion of Gases. — The motion of a gas escaping through a tube is opposed by the friction or viscosity of the gas, and the same is true when the escape is through an aperture so narrow compared with its length that it may be regarded as a tube. But when the escape is through an aperture in a thin wall, the effect of viscosity is very small and may for many purposes be neglected.

In this case the kinetic energy gained by the escaping gas is equal to the work done by the pressure in the vessel in causing the outflow. It would not affect the motion and it will simplify the problem if we suppose that a frictionless tube of the same cross-section as the aperture is connected to the aperture, and that the escaping gas drives a weightless piston along the tube. If the piston moves from B to C in a second, BC equals the velocity v of the escaping gas, and if s is the cross-section of the tube and aperture, the mass of gas that escapes in a second is $BC \cdot s \cdot \rho$ or $vs\rho$, and its kinetic energy is $\frac{1}{2} \cdot vs\rho \cdot v^2$. The work done by the pressure p that causes the outflow (that is, the excess of the pressure in the vessel over the external pressure) is $psBC$ or psv. Equating the work done and the kinetic energy gained, we get

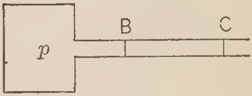

Fig. 117.

$$v = \sqrt{\frac{2p}{\rho}}.$$

Hence under equal pressures the velocities of escape of different gases are inversely as the square roots of their densities. This is the basis of Bunsen's method of comparing the densities of gases.

If the pressure p were supposed due to the weight of a column of the gas of uniform density ρ and height h, we would have $p = g\rho h$. If this be substituted in the above formula, it will be identical with the formula for the outflow of a liquid (§ 150).

180. Diffusion of Gases. — If two equal bottles containing different gases be placed mouth to mouth, each gas

will pass into the other at a very rapid rate, and after a short time each gas will be equally divided between the two bottles. The result is independent of gravity and is the same whether the bottle containing the heavier gas (*e.g.* carbonic acid) is above or below that containing the lighter gas (*e.g.* hydrogen). This process of diffusion accounts for the fact that the proportions of oxygen and nitrogen in the atmosphere are practically the same everywhere.

Diffusion also takes place when two gases are separated by a porous partition such as unglazed earthenware. Lighter gases pass more rapidly than heavier gases through such a partition, but the final result is the same as if the partition were absent. If one end of a glass tube be sealed into a small dry earthenware jar (such as is used in a Bunsen's battery) while the other end is immersed in water in a beaker, and if the jar be covered by an inverted beaker into which coal gas is allowed to stream, air will be forced out through the lower end of the tube, owing to the lighter gas entering through the porous jar faster than the air escapes through it. When the jar has thus become full of a mixture of air and gas, if the large beaker be removed so that the porous jar is now surrounded by air, water will rise in the tube, owing to the gas within the jar escaping more rapidly than the air enters. The difference in the rates of diffusion of different gases through a porous partition is the basis of a method of separating mixed gases.

Fig. 118.

REFERENCES

Poynting and Thomson's "Properties of Matter."

Gray's "Treatise on Physics," Vol. I.

Tait's "Properties of Matter."

"The Laws of Gases" (the original papers of Boyle and Amagat), in "Scientific Memoirs" series.

Risteen's "Molecules and the Molecular Theory of Matter."

Boys's "Soap Bubbles."

PROBLEMS

(The student is recommended to use C. G. S. absolute units when the problem is stated in metric units and F. P. S. gravitational units when the problem is stated in British units.)

1. Find the magnitude and direction of the resultant of two displacements of magnitude 12 and 15 in directions that differ by 30°.

2. Find by the analytical method the magnitude and direction of the resultant of three velocities, 10 east, 20 north, and 16 southwest.

3. A body starts with a velocity of 20 ft. per second and has an acceleration of 32 ft. per second^2 in the direction of motion. What is its velocity and distance from the starting-point 1, 3, and 6 sec. after starting?

4. A body starts with a velocity of 50 cm. per second, and in $6\frac{1}{2}$ sec. has acquired a velocity of 102 cm. per second. What is its acceleration and how far has it travelled?

5. A steamship is moving due east with a velocity of 20 mi. an hour, and to the passengers the wind seems to blow from the north with a velocity of 12 mi. an hour. Find the actual velocity and direction of the wind.

6. A train having a speed of 70 km. per hour is brought to rest in a distance of 600 m. What is its acceleration?

7. A body slides down a smooth inclined plane and in the third second travels 110 cm. What is the inclination of the plane?

8. A carriage wheel, 1 m. in diameter, makes 200 revolutions per minute. What is the instantaneous speed of a point on the tire (1) when it is 1 m. from the ground; (2) when it is 0.5 m. from the ground; (3) when it is on the ground?

9. A body is projected at an angle of 60° with the horizontal with a velocity of 40 m. per sec. How long will it move and how high will it rise? When and where will it again meet the horizontal plane through the starting-point?

10. A railway train rounds a curve of 1000 ft. radius, with a speed of 50 miles per hour. What is its acceleration?

11. How much is the acceleration of a falling body at the equator decreased by the rotation of the earth (assume the radius of the earth to be 4000 mi.)?

12. A flywheel, making 10 revolutions per second, comes to rest in 1 min. Find its angular acceleration and the number of revolutions.

13. Express 980 cm. per second^2 in kilometres per minute.^{2}

14. A force of 1000 dynes acts on a mass of 10 g. for 1 min. Find the velocity acquired and the distance traversed.

15. In what time will a force of 5 kg. weight moves a mass of 10 kg. a distance of 50 m.? What will be the velocity at the end of 10 sec.?

16. What force must act on a mass of 50 kg. to increase its velocity from 100 cm. per second to 200 cm. per second while the body passes over 50 m.?

17. Find the resistance when a body weighing 20 oz., projected along a rough table with a velocity of 48 ft. per second, is brought to rest in 5 sec.

18. What constant force will lift a mass of 50 lb. 200 ft. vertically in 10 sec.? Find the velocity at the end of that time.

19. What pressure will a man who weighs 70 kg. exert upon an elevator descending with an acceleration of 100 cm. per second^2? If ascending with the same acceleration?

20. One minute after leaving a station a train has a velocity of 30 mi. an hour, what is the ratio of the resultant horizontal force to the weight of the train?

21. A train is moving at a rate of 20 mi. an hour when the steam is shut off. If the resistance of friction amounts to $\frac{1}{70}$ of the weight of the train, how far will it run up a 5° incline?

22. A body weighing 2 kg. rests on a table and is acted on by a force of 8 kg. weight, making an angle of 40° with the horizontal. What is the total pressure on the table?

23. The diameter of the bore of a gun is 10 in. and the explosion of the powder exerts a pressure of 30,000 lb. weight per square inch on the end of a projectile which weighs 372 lb. If the pressure of the powder is constant, and the projectile moves to the muzzle in $\frac{1}{100}$ of a second, what is the velocity of the projectile?

24. An iron ball of 40 kg. mass falls 100 cm. vertically and drives a nail 2 cm. into a plank. What is the pressure on the nail if it be supposed constant?

25. A mass of 5 kg. rests on an inclined plane which has a length of 30 cm. and a height of 2 cm. Find the pressure on the plane and the resistance of friction.

26. The ends of a cord 15 ft. long are attached to two pegs at the same level and 10 ft. apart. If a mass of 100 lb. is attached to the middle of the cord, what is the force on each peg?

27. A weight of 100 lb. hangs at the end of a cord. What horizontal force will deflect the cord 30°, and what will be the tension in the cord?

28. A rapid-firing gun delivers in a second 10 projectiles of 1 lb. each with a speed of 2000 ft. per second. What force is required to hold the gun at rest?

29. A baseball weighing 12 oz. and moving with a velocity of 50 ft. per second is struck squarely by a bat and given a velocity of 100 ft. per second in the opposite direction. If the contact lasts .005 sec., what is the average force?

30. If the train in example 10 weighs 500 T., what is the total outward pressure on the rails?

31. A skater describes a circle of 10 m. radius at a speed of 5 m. per second. With what force do his skates press on the ice?

32. A rod of 10 kg. mass and 100 cm. in length revolves about an axis through one end, making 10 revolutions per second. Find the pull on the axis.

33. A falling mass of 200 g. is connected by a string to a mass of 1800 g. lying on a smooth horizontal table. Find the acceleration and the tension of the string.

34. At the foot of a hill, a toboggan has a velocity of 20 ft. per second. If it slides 120 ft. on the horizontal, what is the coefficient of friction?

35. Reduce a force of 20 lb. weight to dynes.

36. A cord passes over two pulleys and through a third movable pulley between them, and is vertical where not in contact with the pulleys. To one end of the cord a mass of 20 kg. is attached and to the other a mass of 10 kg. and the movable weighs 5 kg. Find the accelerations of the masses.

37. What is the period of vibration of a mass of 1 kg. attached to a spiral spring if an additional mass of 100 g. stretches the spring 0.3 cm. farther?

38. A man presses a tool on a grindstone of 1 m. diameter with a force of 10 kg. weight. If the coefficient of friction is 0.2, what force at the end of a crank arm 40 cm. in length will turn the stone?

39. A disk of 500 g. mass and 20 cm. in diameter acquires in 10 sec. a linear velocity of 30 m. per second, in the direction of its axis and an angular velocity of 2 rotations per second about its axis. What forces acted on it?

40. Find the centre of mass of 20, 30, 24, and 60 g. at the corners of a square.

41. Out of a circular disk 16 cm. in diameter a circle 12 cm. in diameter and tangential to the larger is cut. Where is the centre of mass of the remainder?

42. Two cylinders of the same material, each 20 cm. in length and 12 and 6 cm. in diameter respectively, are joined so that their axes coincide. Find the centre of mass of the whole.

43. An iron cylinder 30 cm. in diameter and of 5 kg. mass rolls down a plane 20 ft. long inclined at 30° to the horizontal. What linear velocity does it acquire?

44. Find the resultant of parallel forces 20, 40, and -30 applied at the corners of an equilateral triangle of 10 cm. side.

45. A body is moved from rest without friction by a force that increases uniformly with the distance traversed from 10 lb. weight to 80 lb. weight. Draw a diagram of work done and find the kinetic energy acquired, the total distance traversed being 10 ft.

46. A spiral spring is attached to a 50 kg. weight. What work is done if the increase of length of the spring is 20 cm. when the weight is just lifted?

47. A lever 20 in. long is used to turn a screw with a pitch of $\frac{1}{5}$ in. If a force of 80 lb. weight is applied to the lever, what force will the screw exert?

48. Two uniform beams each 24 ft. long and of 100 lb. mass are in contact at their upper ends, while their lower ends rest on two vertical walls of the same height and 36 ft. apart. Find the horizontal thrust on each wall.

49. A runner has a record of 10 sec. for 100 yd., $6\frac{2}{5}$ sec. for 60 yd., and $4\frac{3}{5}$ sec. for 40 yd. What can be deduced as to the horse-power at which he works in running, if he weighs 140 lb.?

50. A runner can run 100 yards on the horizontal in 10 sec. and the same distance uphill with a rise of 32 ft. in 17.5 sec. At what horse-power does he work, if his weight is 145 lb.?

51. A cable 100 m. long and of 50 lb. mass hangs vertically from a viaduct. How much work will be expended in raising it?

52. If the connection of the rod to its axis in problem 32 should break, how would the rod move?

53. A grindstone weighs 75 kg. and has a diameter of 1 m. How much energy is stored in it when it makes 300 revolutions per minute?

54. When a hoop rolls on a rough plane, what is the ratio of kinetic energy of rotation to that of translation?

55. Calculate the activity of an engine that raises 1,000,000 gal. (each = 10 lb.) of water in 8 hr. from a depth of 125 ft.

56. The top of a table a metre square projects 5 cm. beyond the legs. If the table weighs 10 kg., what weight hung from a corner will overturn it?

57. The distance between the centre of the moon and that of the earth is 60 times the radius of the earth, and the mass of the earth is 82 times that of the moon. Find their centre of mass.

58. What is the horse-power of a locomotive that gives a train of 200 T. a velocity of 30 mi. an hour in a distance of 1000 ft. up an incline of 1 in 1000, the total resistance of friction being 15 lb. weight per ton?

59. What is the period of vibration of a disk 20 cm. in diameter suspended on a horizontal axis perpendicular to the disk and attached to the rim?

60. What is the period of vibration of a uniform rod 1 m. long about a horizontal axis 10 cm. from one end? About what other points is the period of vibration the same?

61. The area of the "water-line" of a ship is 3000 sq. ft. What depth will the ship sink in fresh water if 100 T. be placed in it?

62. How much will the above-mentioned vessel rise when it passes into salt water of density 1.026?

63. The density of a body is 2 and in air of density .0013 it weighs 100.00 g. What is its true weight, the density of the weights being 9?

64. When carried from the ground to the roof of a building a barometer falls 1.5 mm. What is the height of the building?

65. A mass of copper, suspected of being hollow, weighs 523 g. in air and 447.5 g. in water. What is the volume of the cavity?

66. The specific gravity of ice is .918 and that of sea water 1.026. What is the total volume of an iceberg of which 700 cu. yd. is exposed?

67. A block of wood weighing 1 kg., the density of which is 0.7, is to be loaded with lead so as to float with 0.9 of its volume immersed. What weight of lead is required (1) if the lead is immersed? (2) if it is not immersed?

68. A body A weighs 7.55 g. in air, 5.17 g. in water, and 6.35 g. in a liquid B. Find the density of A and that of B.

69. A retaining wall 3 m. wide and 40 m. long is inclined at 40° to the horizontal. Find the total pressure against it in kilogrammes weight when the water rises to the top.

70. How far will water be projected horizontally from an aperture 3 m. below the level of water in a tank and 10 m. above the ground?

71. The surface tension of a soap-bubble solution is 27.45 dynes per cm. How much greater is the pressure inside a soap-bubble of 3 cm. radius than in the outer air?

72. If a submarine boat weighed 50 tons and displaced 3000 cu. ft. when immersed, how much water would it have to take in to sink?

73. A cylindrical diving-bell 2 m. in height is lowered until the top of the bell is 6 m. below the surface of the water. How high will the water rise in the bell if the height of the barometer is 76 cm.? What air pressure in the bell would keep the water out?

74. A Fortin barometer reads 73 cm. at a point 150 m. above sea-level in latitude 41°, at a temperature of 21° C. Reduce the reading to 0° C. at sea-level in latitude 45°.

75. A wire 300 mm. long and 1 mm. in diameter is stretched 1 mm. by a weight of 3000 g. Calculate Young's modulus.

76. To a wire 100 cm. long and 0.24 mm. in diameter a disk whose moment of inertia is 400 g. cm.^{2} is attached. The period of torsional vibrations is 8 sec. Calculate the shear modulus.

77. A piece of shafting 10 m. long and of 5 cm. radius is twisted 1° by a certain moment. How may the shaft be changed so that the twist will be 30′?

TABLES

CONVERSION TABLE

1 cm.	= 0.3937 in.	1 inch	= 2.540 cm.
1 sq. cm.	= 0.1550 sq. in.	1 sq. in.	= 6.451 sq. cm.
1 cc.	= 0.0610 cu. in.	1 cu. in.	= 16.386 cc.
1 kg.	= 2.205 lbs.	1 lb.	= 435.6 gm.
1 gal.	= 4543 cc.	1 litre	= 1.7608 pints

ACCELERATION OF FALLING BODY

(In C. G. S. units)

Boston	980.382	Philadelphia	980.182
Chicago	980.264	San Francisco	979.951
Cincinnati	979.990	St. Louis	979.987
Cleveland	980.227	Terre Haute	980.058
Denver	979.595	Washington	980.100

Berlin	981.240	Paris	980.960
(Equator	978.070)	(Pole	983.110)
Greenwich	981.170	Rome	980.310
Hammerfest	982.580	Vienna	980.852

DENSITY

Aluminium	2.60	Iron (cast)	7.40
Brass (about)	8.50	Iron (wrought)	7.86
Copper	8.92	Lead	11.30
Gold	19.30	Platinum	21.50
		Silver	10.53

ELASTIC CONSTANTS

(Rough averages; in C. G. S. units)

	Shear Modulus	Young's Modulus	Bulk Modulus	Parson's Ratio
Copper	4×10^{11}	11×10^{11}	17×10^{11}	.30
Glass	2×10^{11}	6×10^{11}	4×10^{11}	.23
Iron (wrought)	7×10^{11}	19×10^{11}	15×10^{11}	.30
Lead	$.2 \times 10^{11}$	1×10^{11}	4×10^{11}	.37
Steel	8×10^{11}	23×10^{11}	17×10^{11}	.29

VISCOSITY

(In C. G. S. units; at 20° C.)

Alcohol	0.0011	Glycerine	8.0
Ether	0.0026	Water	0.010

SURFACE TENSION

(In C. G. S. units; at 20° C.)

Alcohol	21	Mercury	450
Ether	17	Water	74

ANGLE OF CONTACT

Alcohol	0°	Mercury (about)	145°
Ether	16°	Water (about)	0°

TRIGONOMETRICAL RATIOS

Angle	Radians	Sine	Tangent	Cotangent	Cosine		
0°	0	0	0	∞	1	1.5708	90°
1	.0175	.0175	.0175	57.2900	.9998	1.5533	89
2	.0349	.0349	.0349	28.6363	.9994	1.5359	88
3	.0524	.0523	.0524	19.0811	.9986	1.5184	87
4	.0698	.0698	.0699	14.3006	.9976	1.5010	86
5	.0873	.0872	.0875	11.4301	.9962	1.4835	85
6	.1047	.1045	.1051	9.5144	.9945	1.4661	84
7	.1222	.1219	.1228	8.1443	.9925	1.4486	83
8	.1396	.1392	.1405	7.1154	.9903	1.4312	82
9	.1571	.1564	.1584	6.3138	.9877	1.4137	81
10	.1745	.1736	.1763	5.6713	.9848	1.3963	80
11	.1920	.1908	.1944	5.1446	.9816	1.3788	79
12	.2094	.2079	.2126	4.7046	.9781	1.3614	78
13	.2269	.2250	.2309	4.3315	.9744	1.3439	77
14	.2443	.2419	.2493	4.0108	.9703	1.3265	76
15	.2618	.2588	.2679	3.7321	.9659	1.3090	75
16	.2793	.2756	.2867	3.4874	.9613	1.2915	74
17	.2967	.2924	.3057	3.2709	.9563	1.2741	73
18	.3142	.3090	.3249	3.0777	.9511	1.2566	72
19	.3316	.3256	.3443	2.9042	.9455	1.2392	71
20	.3491	.3420	.3640	2.7475	.9397	1.2217	70
21	.3665	.3584	.3839	2.6051	.9336	1.2043	69
22	.3840	.3746	.4040	2.4751	.9272	1.1868	68
23	.4014	.3907	.4245	2.3559	.9205	1.1694	67
24	.4189	.4067	.4452	2.2460	.9135	1.1519	66
25	.4363	.4226	.4663	2.1445	.9063	1.1345	65
26	.4538	.4384	.4877	2.0503	.8988	1.1170	64
27	.4712	.4540	.5095	1.9626	.8910	1.0996	63
28	.4887	.4695	.5317	1.8807	.8830	1.0821	62
29	.5061	.4848	.5543	1.8040	.8746	1.0647	61
30	.5236	.5000	.5774	1.7321	.8660	1.0472	60
31	.5411	.5150	.6009	1.6643	.8572	1.0297	59
32	.5585	.5299	.6249	1.6003	.8480	1.0123	58
33	.5760	.5446	.6494	1.5399	.8387	.9948	57
34	.5934	.5592	.6745	1.4826	.8290	.9774	56
35	.6109	.5736	.7002	1.4281	.8192	.9599	55
36	.6283	.5878	.7265	1.3764	.8090	.9425	54
37	.6458	.6018	.7536	1.3270	.7986	.9250	53
38	.6632	.6157	.7813	1.2799	.7880	.9076	52
39	.6807	.6293	.8098	1.2349	.7771	.8901	51
40	.6981	.6428	.8391	1.1918	.7660	.8727	50
41	.7156	.6561	.8693	1.1504	.7547	.8552	49
42	.7330	.6691	.9004	1.1106	.7431	.8378	48
43	.7505	.6820	.9325	1.0724	.7314	.8203	47
44	.7679	.6947	.9657	1.0355	.7193	.8029	46
45	.7854	.7071	1.0000	1.0000	.7071	.7854	45
		Cosine	Cotangent	Tangent	Sine	Radians	Angle

LOGARITHMS

	0	1	2	3	4	5	6	7	8	4	1 2 3	4 5 6	2 8 1
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4 8 12	17 21 25	29 33 37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4 8 11	15 19 23	26 30 34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3 7 10	14 17 21	24 28 31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3 6 10	13 16 19	23 26 29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3 6 9	12 15 18	21 24 27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3 6 8	11 14 17	20 22 25
16	2041	2068	2095	2122	2148	2175	2201	2227	1253	2279	3 5 8	11 13 16	18 21 24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2 5 7	10 12 15	17 20 22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2 5 7	9 12 14	16 19 21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2 4 7	9 11 13	16 18 20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2 4 6	8 11 13	15 17 19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2 4 6	8 10 12	14 16 18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2 4 6	8 10 12	14 15 17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2 4 6	7 9 11	13 15 17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2 4 5	7 9 11	12 14 16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2 3 5	7 9 10	12 14 15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2 3 5	7 8 10	11 13 15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2 3 5	6 8 9	11 13 14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2 3 5	6 8 9	11 12 14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1 3 4	6 7 9	10 12 13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1 3 4	6 7 9	10 11 13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1 3 4	6 7 8	10 11 12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1 3 4	5 7 8	9 11 12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1 3 4	5 6 8	9 10 12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1 3 4	5 6 8	9 10 11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1 2 4	5 6 7	9 10 11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1 2 4	5 6 7	8 10 11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1 2 3	5 6 7	8 9 10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1 2 3	5 6 7	8 9 10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1 2 3	4 5 7	8 9 10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1 2 3	4 5 6	8 9 10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1 2 3	4 5 6	7 8 9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1 2 3	4 5 6	7 8 9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1 2 3	4 5 6	7 8 9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1 2 3	4 5 6	7 8 9
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46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1 2 3	4 5 6	7 7 8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1 2 3	4 5 5	6 7 8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1 2 3	4 4 5	6 7 8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1 2 3	4 4 5	6 7 8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1 2 3	3 4 5	6 7 8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1 2 3	3 4 5	6 7 8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1 2 2	3 4 5	6 7 7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1 2 2	3 4 5	6 6 7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1 2 2	3 4 5	6 6 7

LOGARITHMS

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56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
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88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
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90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
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92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

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